Cuaderno de notas de trabajo

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Cuaderno 45

Páginas 46. → 51 Ecuación diferencial de las líneas de universo de una partícula exp de campo central en forma invariante.

pág. 43 Ecuaciones diferenciales de las líneas de universo de una partícula exploradora en el campo central. Forma explícita.

El Lagrangiano del campo central

1) $L = \frac{1}{2} e^{\frac{2M}{r}} (1 + v^2)$

sugiere la existencia de un Lagrangiano general para las órbitas de un campo arbitrario en la Teoría de Birkhoff.

Ensayemos un Lagrangiano producto de e^F y de G en donde F y G son funciones de la velocidad $\vec{v}$ y de la posición $\vec{r}$ de la partícula.

Usamos la notación:

$$\frac{\partial F}{\partial \vec{v}} = \left(\frac{\partial F}{\partial v^x}, \frac{\partial F}{\partial v^y}, \frac{\partial F}{\partial v^z} \right)$$

Entonces:

2) $$L = e^F G$$

3) Las ecuaciones de Lagrange son entonces:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) = \frac{\partial L}{\partial \vec{r}} = \nabla L .$$

$$L = e^F G$$

4) $$\frac{\partial L}{\partial \vec{v}} = e^F \frac{\partial G}{\partial \vec{v}} + e^F G \frac{\partial F}{\partial \vec{v}}$$

$$\boxed{\frac{\partial L}{\partial \vec{v}} = e^F \left[\frac{\partial G}{\partial \vec{v}} + G \frac{\partial F}{\partial \vec{v}} \right]}$$

5) $$\boxed{\nabla L = e^F \left[\nabla G + G \nabla F \right]}$$

6) $$\cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) = e^F \left[\nabla \frac{\partial G}{\partial \vec{v}} + G \nabla \frac{\partial F}{\partial \vec{v}} + (\nabla G) \right.}$$

$$\boxed{\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) = e^F \Big[& \frac{\partial}{\partial t} \frac{\partial G}{\partial \vec{v}} + (\vec{v} \cdot \nabla) \frac{\partial G}{\partial \vec{v}} \\ & + \frac{\partial G}{\partial t} \frac{\partial F}{\partial \vec{v}} + G \frac{\partial}{\partial t} \frac{\partial F}{\partial \vec{v}} \\ & + \left\{ (\vec{v} \cdot \nabla) G \right\} \frac{\partial F}{\partial \vec{v}} + G \left\{ (\vec{v} \cdot \nabla) \frac{\partial F}{\partial \vec{v}} \right\} \Big] \end{aligned}}$$

7) <u>Ecuaciones de Lagrange</u>:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) - \frac{\partial L}{\partial \vec{r}} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) - \nabla L = 0$$

$$\boxed{\left.\begin{aligned}&\frac{\partial}{\partial t}\frac{\partial G}{\partial \vec{v}} + (\vec{v}\cdot\nabla)\frac{\partial G}{\partial \vec{v}} + \frac{\partial G}{\partial t}\frac{\partial F}{\partial \vec{v}} + G\frac{\partial}{\partial t}\frac{\partial F}{\partial \vec{v}} \\ &+\frac{\partial F}{\partial \vec{v}}\left\{(\vec{v}\cdot\nabla)G\right\} + G\left\{(\vec{v}\cdot\nabla)\frac{\partial F}{\partial \vec{v}}\right\} \\ &-\nabla G - G\nabla F\end{aligned}\right\} = 0}$$

Campo Central

$$L = \int_{t_1}^{t_2} e^{\frac{2M}{r}}(1+v^2)\,dt$$

$$\frac{d}{dt}\left[2e^{\frac{2M}{r}}\vec{v}\right] + \frac{2M}{r^2}e^{\frac{2M}{r}}(\nabla r)(1+v^2) = 0$$

$$2e^{\frac{2M}{r}}\vec{a} - \frac{4M}{r^2}\dot{r}\vec{v}e^{\frac{2M}{r}} + \frac{2M}{r^2}e^{\frac{2M}{r}}(\nabla r)(1+v^2) = 0$$

$$\vec{a} - \frac{2M}{r^2}\dot{r}\vec{v} + \cancel{\frac{2M}{r^2}\nabla r}\;\frac{2M}{r^3}\vec{r}(1+v^2) = 0$$

$$\boxed{\vec{a} = \frac{2M}{r^3}\left[-\vec{r}(1+v^2) + 2r\dot{r}\vec{v}\right]}$$

$$\cancel{\frac{1}{?}\frac{\partial}{\partial t}\frac{\partial G}{\partial \vec{v}}\partial t(\vec{v}\cdot\nabla)}$$

$$\cancel{G = v^2 + \vec{V}\cdot\vec{v} + \frac{1}{2}\{(T\vec{v})\cdot\vec{v}\}}$$

Escalares significativos

$$v^2,\ \vec{V}\cdot\vec{v},\ (T\vec{v})\cdot\vec{v},\ \Phi,\ \cancel{\Phi v^2},\ \Phi v^2$$

$$G = a_1 v^2 + a_2 \vec{V}\cdot\vec{v} + a_3 (T\vec{v}\cdot\vec{v}) + a_4 \Phi + a_5 \Phi v^2$$

$$F = b_1 v^2 + b_2 \vec{V}\cdot\vec{v} + b_3 (T\vec{v}\cdot\vec{v}) + b_4 \Phi + b_5 \Phi v^2$$

$$\boxed{\begin{aligned}\frac{\partial G}{\partial \vec{v}} &= 2a_1\vec{v} + a_2\vec{V} + 2a_3(T\vec{v}) + 2a_5\Phi\vec{v}\\ \frac{\partial F}{\partial \vec{v}} &= 2b_1\vec{v} + b_2\vec{V} + 2b_3(T\vec{v}) + 2b_5\Phi\vec{v}\end{aligned}}$$

$$L = \frac{1}{2}(\text{exponencial})\{2\vec{V}\cdot\vec{v} + (T\vec{v})\cdot\vec{v} + v^2\}$$

$$L = \frac{1}{2}(\text{exponencial})\{v^2 + (T\vec{v})\cdot\vec{v} + 2\vec{V}\cdot\vec{v}\}$$

$$L = \frac{1}{2}\left[\exp\{\Phi + a_1(\vec{V}\cdot\vec{v}) + a_2(T\vec{v})\cdot\vec{v}\}\right]\left[v^2 + b_1(\vec{V}\cdot\vec{v}) + b_2(T\vec{v})\cdot\vec{v}\right]$$

Campo Central

$$h_{ij} = \begin{pmatrix} \frac{M}{r} & 0 & 0 & 0 \\ 0 & \frac{M}{r} & 0 & 0 \\ 0 & 0 & \frac{M}{r} & 0 \\ 0 & 0 & 0 & \frac{M}{r} \end{pmatrix}$$

$$h_{ij} v^i v^j = \frac{M}{r} \frac{1+v^2}{1-v^2}$$

$$h_{ij} v^i v^j dt = \frac{M}{r} \frac{1+v^2}{1-v^2} dt$$

$$\frac{M}{r} =$$

$$\Phi + 2\vec{V}\cdot\vec{v} + (T\vec{v})\cdot\vec{v} = A$$

$$\frac{\Phi + 2\vec{V}\cdot\vec{v} + (T\vec{v})\cdot\vec{v}}{\Phi} = \frac{A}{\Phi}$$

$$e^{2\Phi}\left[\frac{\Phi + 2\vec{V}\cdot\vec{v} + (T\vec{v})\cdot\vec{v}}{\Phi}\right]$$

Lagrangiano parcial.

$$\mathcal{L} = \tfrac{1}{2}v^2 + \Phi + 2\vec{V}\cdot\vec{v} + (T\vec{v})\cdot\vec{v}$$

$$\frac{\partial\mathcal{L}}{\partial\vec{v}} = \vec{v} + 2\vec{V} + 2(T\vec{v})$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\vec{v}}\right) = \left[\vec{a} + 2\frac{\partial\vec{V}}{\partial t} + 2(\vec{v}\cdot\nabla)\vec{V} + 2\frac{\partial T}{\partial t}\vec{a} + 2(\vec{v}\cdot\nabla)(T\vec{v}) + 2(T\vec{a})\right]$$

$$\nabla\mathcal{L} = \nabla\left[\Phi + 2\vec{V}\cdot\vec{v} + (T\vec{v})\cdot\vec{v}\right]$$

$$\vec{a} + 2(T\vec{a})$$

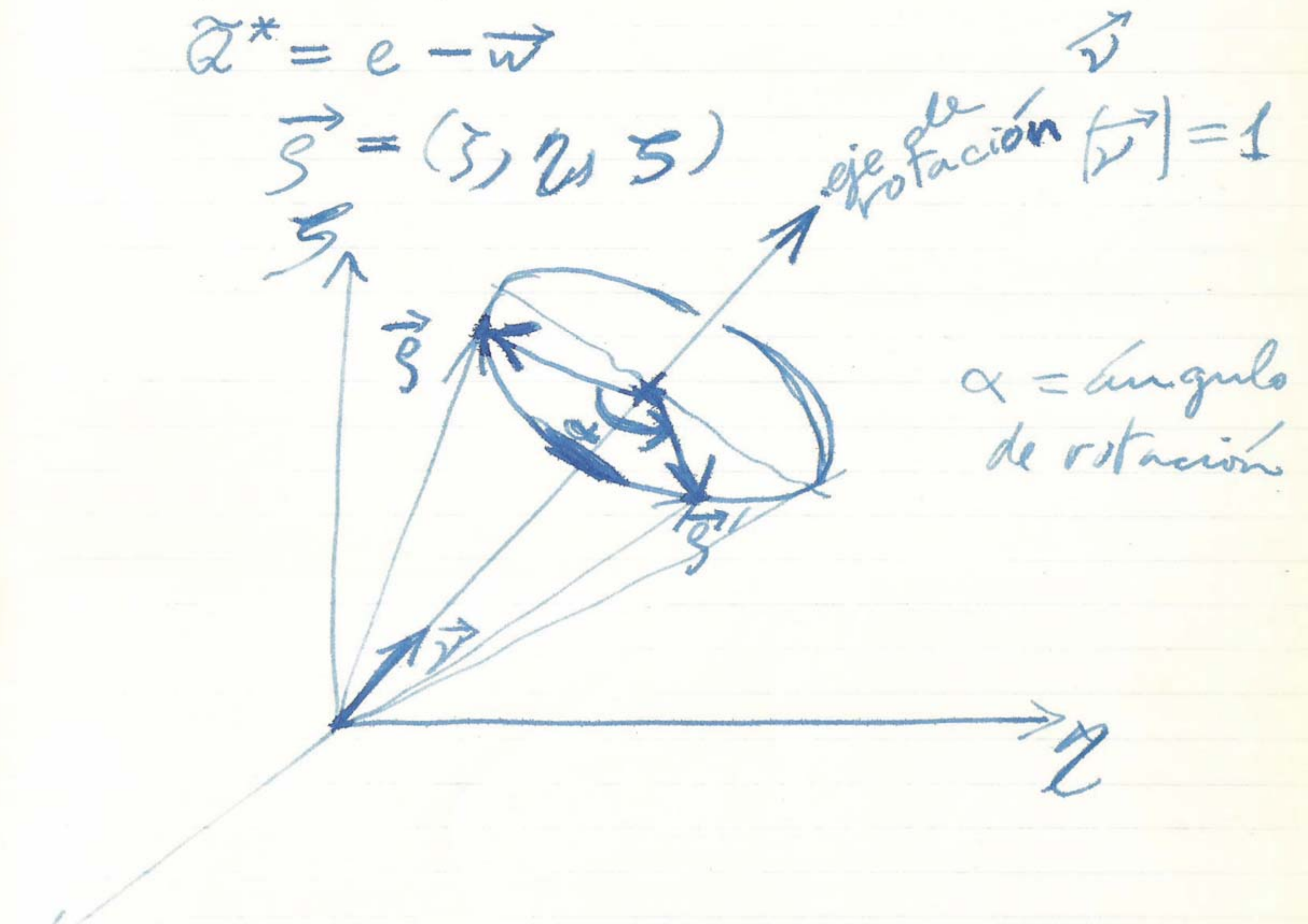
Cuerpo rígido

$\tilde{Q}$ --- cuaternión.

$\tilde{Q} = e + \vec{w}$

$\tilde{Q}^* = e - \vec{w}$

$\vec{\xi} = (\xi, \eta, \zeta)$



$\vec{\xi}$ --- antes de la rotación

$\vec{\xi}'$ --- despuest de la rotación

$\vec{\rho}$ $\vec{\rho}$

Componente de $\vec{\rho}$ a lo largo del eje de rotación } $\vec{\rho}\cdot\vec{\nu}$

Como vector $(\vec{\rho}\cdot\vec{\nu})\vec{\nu}$

Vector componente de $\vec{\rho}$ ⊥ al eje de rotación } $\vec{\rho}-(\vec{\rho}\cdot\vec{\nu})\vec{\nu}$

Vector ⊥ a $\vec{\nu}$ y a $\vec{\rho}$ } $\vec{\nu}\times\vec{\rho}$

$$(\vec{\nu}\times\vec{\rho})^2=(\vec{\nu}\cdot\vec{\nu})(\vec{\rho}\cdot\vec{\rho})-(\vec{\nu}\cdot\vec{\rho})^2$$

$$(\vec{\nu}\times\vec{\rho})^2=\rho^2-(\vec{\nu}\cdot\vec{\rho})^2$$

Vector unitario ⊥ a $\vec{\nu}$ y a $\vec{\rho}$ } $\dfrac{\vec{\nu}\times\vec{\rho}}{\sqrt{\rho^2-(\vec{\nu}\cdot\vec{\rho})^2}}$

Vector unitario ⊥ a $\vec{\rho}$ y al eje de rotación }

Magnitud de $\vec{\rho}-(\vec{\rho}\cdot\vec{\nu})\vec{\nu}$

$$[\vec{\rho}-(\vec{\rho}\cdot\vec{\nu})\vec{\nu}]^2=$$

$$\left[\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\,\vec{\nu}\right]^2 = \rho^2 + (\vec{\rho}\cdot\vec{\nu})^2 - 2(\vec{\rho}\cdot\vec{\nu})^2$$

$$= \rho^2 - (\vec{\rho}\cdot\vec{\nu})^2$$

~~vector unitario $\perp$ a $\vec{\rho}$ y al eje de rotación~~

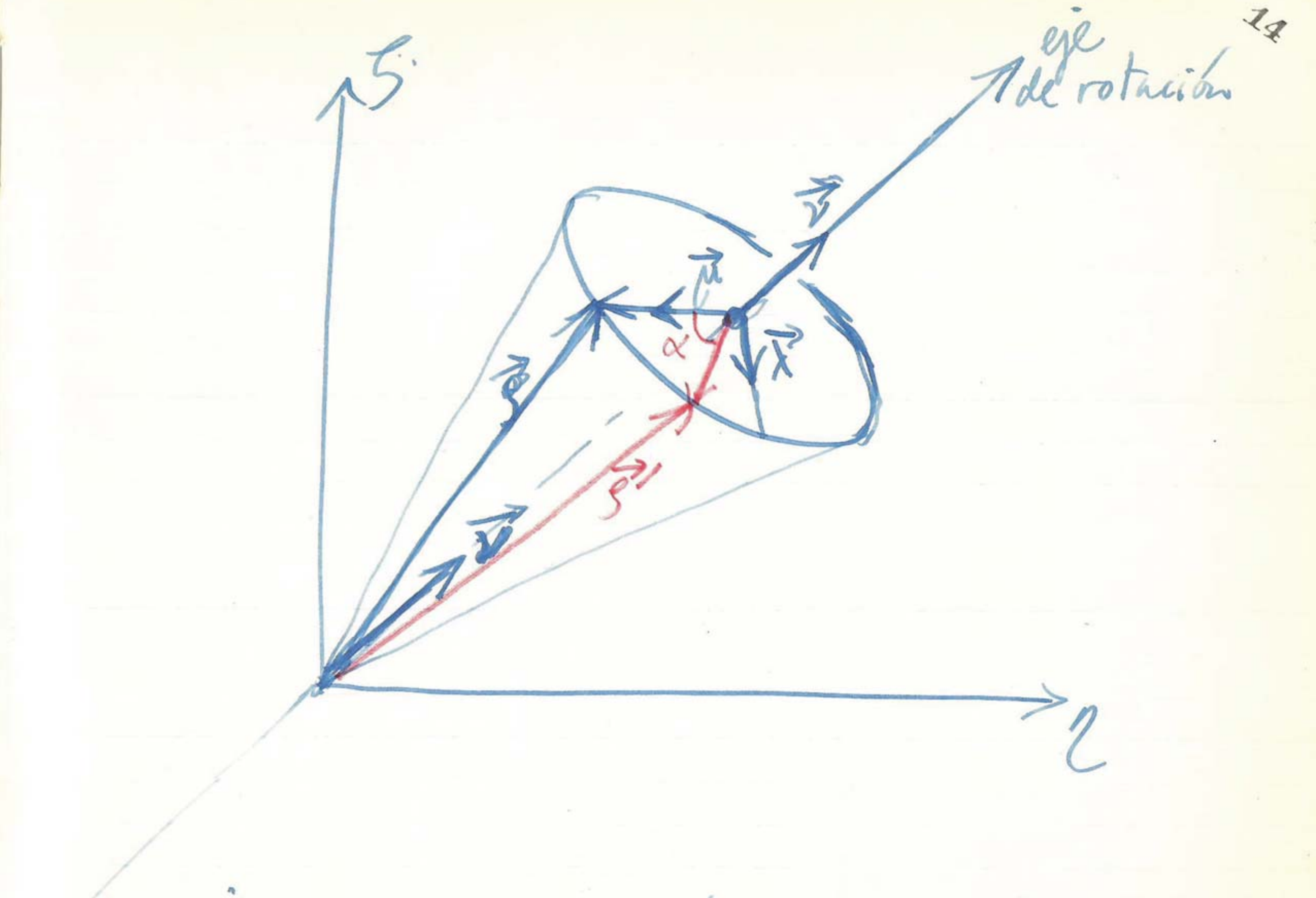
Vector unitario $\perp$ a $\vec{\nu}$ y a lo largo de la componente de $\vec{\rho} \perp$ a $\vec{\nu}$

$$\left.\right\}\ \frac{\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\,\vec{\nu}}{\sqrt{\rho^2 - (\vec{\rho}\cdot\vec{\nu})^2}}$$

$$\vec{\rho}\,' = (\vec{\rho}\cdot\vec{\nu})\,\vec{\nu} + \frac{\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\,\vec{\nu}}{\cancel{\sqrt{\rho^2 - (\vec{\rho}\cdot\vec{\nu})^2}}}\cos\alpha + \frac{\cancel{\vec{\rho}}\ \vec{\nu}\times\vec{\rho}}{\cancel{\sqrt{\rho^2 - (\vec{\rho}\cdot\vec{\nu})^2}}}\,\mathrm{sen}\,\alpha$$

$$\boxed{\vec{\rho}\,' = (\vec{\rho}\cdot\vec{\nu})\,\vec{\nu} + \frac{\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\,\vec{\nu}}{\sqrt{\rho^2 - (\vec{\rho}\cdot\vec{\nu})^2}}\cos\alpha + \frac{\vec{\nu}\times\vec{\rho}}{\sqrt{\rho^2 - (\vec{\rho}\cdot\vec{\nu})^2}}\,\mathrm{sen}\,\alpha}$$

Para comprobación calcúlese $(\vec{\rho}\,')^2$ tiene que ser igual a $\vec{\rho}^{\,2}$



$\vec{\nu}$ --- vector unitario a lo largo del eje de rotación

$\vec{\mu}$ --- vector unitario $\perp$ al eje de rotación, a lo largo de la componente de $\vec{\rho}$ $\perp$ al eje de rotación.

$\vec{\lambda}$ --- vector unitario $\perp$ a $\vec{\nu}$ y a $\vec{\mu}$.

Vector componente de $\vec{\rho}$ a lo largo del eje de rotación $\Big\}$ $(\vec{\rho}\cdot\vec{\nu})\,\vec{\nu}$

Vector componente de $\vec{\rho}$ $\perp$ al eje de rotación $\Big\}$ $\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}$

$$\vec{\mu} = \frac{\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}}{|\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}|}$$

$$[\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}]^2 = \rho^2 + (\vec{\rho}\cdot\vec{\nu})^2 - 2(\vec{\rho}\cdot\vec{\nu})^2$$

$$[\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}]^2 = \rho^2 - (\vec{\rho}\cdot\vec{\nu})^2$$

$$|\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}| = \left|\sqrt{\rho^2 - (\vec{\rho}\cdot\vec{\nu})^2}\right| = R$$

$$\boxed{\vec{\mu} = \frac{\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}}{\sqrt{\rho^2 - (\vec{\rho}\cdot\vec{\nu})^2}}}$$

$$\vec{\lambda} = \vec{\nu} \times \vec{\mu}$$

$$\boxed{\vec{\lambda} = \frac{\vec{\nu} \times \vec{\rho}}{\sqrt{\rho^2 - (\vec{\rho}\cdot\vec{\nu})^2}}}$$

Componente (escalar) de $\vec{\rho}$ $\perp$ al eje de rotación $\Big\}$ $\left|\sqrt{\rho^2 - (\vec{\rho}\cdot\vec{\nu})^2}\right| = R$

Vector Componente de $\vec{\rho}'$ $\perp$ al eje de rotación $\left.\right\}$ $R\cos\alpha\,\vec{\mu} + R\,\text{sen}\,\alpha\,\vec{\lambda}$

$$\vec{\mu} = \frac{\left[\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\,\vec{\nu}\right]}{R}$$

$$\vec{\lambda} = \frac{\vec{\nu}\times\vec{\rho}}{R}$$

Vector componente de $\vec{\rho}'$ $\perp$ al eje de rotación $\left.\right\}$ $\left[\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\,\vec{\nu}\right]\cos\alpha + (\vec{\nu}\times\vec{\rho})\,\text{sen}\,\alpha$

$$\boxed{\vec{\rho}\,' = \left[\left[\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\,\vec{\nu}\right]\cos\alpha + \left(\vec{\nu}\times\vec{\rho}\right]\text{sen}\,\alpha + (\vec{\rho}\cdot\vec{\nu})\,\vec{\nu}\right]}$$

$$\vec{\rho}\,' = \left[(\vec{\rho}\cdot\vec{\nu})\,\vec{\nu} + \left[\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\,\vec{\nu}\right]\cos\alpha + (\vec{\nu}\times\vec{\rho})\,\text{sen}\,\alpha\right]$$

$$\boxed{\vec{\rho}\,' = \left[\vec{\rho}\cos\alpha + (\vec{\rho}\cdot\vec{\nu})\left[1-\cos\alpha\right]\vec{\nu} + (\vec{\nu}\times\vec{\rho})\,\text{sen}\,\alpha\right]}$$

$$\vec{\rho}'^2 = \Big[\underline{\rho^2\cos^2\alpha} + (\vec{\rho}\cdot\vec{\nu})^2\left[1-2\cos\alpha+\cos^2\alpha\right] + \left[\rho^2-(\vec{\rho}\cdot\vec{\nu})^2\right]\operatorname{sen}^2\alpha + 2(\vec{\rho}\cdot\vec{\nu})^2\left[\cos\alpha-\cos^2\alpha\right]\Big]$$

$$\vec{\rho}'^2 = \rho^2 + (\vec{\rho}\cdot\vec{\nu})^2\left[1-2\cos\alpha+\cos^2\alpha-\operatorname{sen}^2\alpha + 2\cos\alpha - 2\cos^2\alpha\right]$$

$$¡¡\ \boxed{\vec{\rho}'^2 = \rho^2}\ !!$$

$$|\vec{\rho}'| = |\vec{\rho}|$$

$$\vec{\rho}'\cdot\vec{\nu} = (\vec{\rho}\cdot\vec{\nu})\left[\cancel{\cos\alpha} + 1 - \cancel{\cos\alpha}\right]$$

$$(\vec{\rho}'\cdot\vec{\nu}) = (\vec{\rho}\cdot\vec{\nu})$$

$$\vec{\rho}' - (\vec{\rho}'\cdot\vec{\nu})\vec{\nu} = \Big[\vec{\rho}\cos\alpha + (\vec{\rho}\cdot\vec{\nu})(1-\cos\alpha)\vec{\nu} + (\vec{\nu}\times\vec{\rho})\operatorname{sen}\alpha - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}\cos\alpha - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}(1-\cos\alpha)\Big]$$

$$\vec{\rho}\,' - (\vec{\rho}\,'\cdot\vec{\nu})\vec{\nu} = \Big[\vec{\rho}\cos\alpha + \vec{\nu}(\vec{\rho}\cdot\vec{\nu})\big[\cancel{1} - \cos\alpha - \cos\alpha \cancel{-1+\cos\alpha}\big] + (\vec{\nu}\times\vec{\rho})\,\mathrm{sen}\,\alpha\Big]$$

$$\boxed{\vec{\rho}\,' - (\vec{\rho}\,'\cdot\vec{\nu})\vec{\nu} = \Big[\vec{\rho}\cos\alpha - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}\cos\alpha + (\vec{\nu}\times\vec{\rho})\,\mathrm{sen}\,\alpha\Big]}$$

$$\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}$$

$$\big[\vec{\rho}\,' - (\vec{\rho}\,'\cdot\vec{\nu})\vec{\nu}\big]\cdot\big[\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}\big] =$$

$$\rho^2\cos\alpha - (\vec{\rho}\cdot\vec{\nu})^2\cos\alpha \cancel{- (\vec{\rho}\cdot\vec{\nu})^2\cos\alpha + (\vec{\rho}\cdot\vec{\nu})^2\cos\alpha}$$

$$\big[\vec{\rho}\,' - (\vec{\rho}\,'\cdot\vec{\nu})\vec{\nu}\big]\cdot\big[\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}\big] =$$

$$\big[\rho^2 - (\vec{\rho}\cdot\vec{\nu})^2\big]\cos\alpha$$

$$\boxed{\vec{\rho}\,' = \left[\vec{\rho}\cos\alpha + (\vec{\rho}\cdot\vec{\nu})[1-\cos\alpha]\vec{\nu} + (\vec{\nu}\times\vec{\rho})\,\mathrm{sen}\,\alpha\right]}$$

$$\tilde{Q} = e + f\vec{\nu}$$
$$\tilde{Q}^* = e - f\vec{\nu}$$

$$(e+f\vec{\nu})\vec{\rho}\,(e-f\vec{\nu}) -$$

$$(e+f\vec{\nu})\vec{\rho} = -f(\vec{\nu}\cdot\vec{\rho}) + e\vec{\rho} + f(\vec{\nu}\times\vec{\rho})$$

$$\left[-f(\vec{\nu}\cdot\vec{\rho}) + \{e\vec{\rho} + f(\vec{\nu}\times\vec{\rho})\}\right][e - f\vec{\nu}] =$$

$$-ef(\vec{\nu}\cdot\vec{\rho}) + ef(\vec{\nu}\cdot\vec{\rho}) = \text{parte escalar} = 0$$

$$+f^2(\vec{\nu}\cdot\vec{\rho})\vec{\nu} + e^2\vec{\rho} + ef(\vec{\nu}\times\vec{\rho})$$

$$-ef(\vec{\rho}\times\vec{\nu}) - f^2[(\vec{\nu}\times\vec{\rho})\times\vec{\nu}]$$

$$(\vec{\nu}\times\vec{\rho})\times\vec{\nu} = [1]\vec{\rho} - [\vec{\nu}\cdot\vec{\rho}]\vec{\nu}$$
$$(\vec{\nu}\times\vec{\rho})\times\vec{\nu} = \vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}$$

$$\tilde{Q}\vec{\rho}\,\tilde{Q}^{*} = \left[(e^2-f^2)\vec{\rho} + 2(f^2)(\vec{\rho}\cdot\vec{\nu})\vec{\nu} + 2ef(\vec{\nu}\cdot\vec{\rho}) \right]$$

$$e^2 - f^2 = \cos\alpha$$

$$2f^2 = 1 - \cos\alpha$$

$$2ef = \operatorname{sen}\alpha$$

$$e = \cos\frac{\alpha}{2} \qquad f = \operatorname{sen}\frac{\alpha}{2}$$

$$e^2 - f^2 = \cos\alpha$$

$$2f^2 = 2\operatorname{sen}^2\frac{\alpha}{2}$$

$$2ef = \operatorname{sen}\alpha$$

$$\tilde{Q} = \cos\frac{\alpha}{2} + \operatorname{sen}\frac{\alpha}{2}\vec{\nu}$$

$$\tilde{Q}^{*} = \cos\frac{\alpha}{2} - \operatorname{sen}\frac{\alpha}{2}\vec{\nu}$$

$$\tilde{Q}\vec{\rho} = -\operatorname{sen}\frac{\alpha}{2}(\vec{\nu}\cdot\vec{\rho}) + \cos\frac{\alpha}{2}\vec{\rho} + \operatorname{sen}\frac{\alpha}{2}(\vec{\nu}\times\vec{\rho})$$

$$\begin{aligned}\tilde{Q}\vec{\rho}\,\tilde{Q}^{*} = \Big[& \cancel{-\operatorname{sen}\frac{\alpha}{2}\cos\frac{\alpha}{2}(\vec{\nu}\cdot\vec{\rho}) + \operatorname{sen}\frac{\alpha}{2}\cos\frac{\alpha}{2}(\vec{\nu}\cdot\vec{\rho})} \\ & + \cos^2\frac{\alpha}{2}\vec{\rho} + \operatorname{sen}\frac{\alpha}{2}\cos\frac{\alpha}{2}(\vec{\nu}\times\vec{\rho}) \\ & + \operatorname{sen}^2\frac{\alpha}{2}(\vec{\nu}\cdot\vec{\rho})\vec{\nu} - \operatorname{sen}\frac{\alpha}{2}\cos\frac{\alpha}{2}(\vec{\rho}\times\vec{\nu}) \\ & - \operatorname{sen}^2\frac{\alpha}{2}\left[(1)\vec{\rho} - (\vec{\rho}\cdot\vec{\nu})\vec{\nu}\right]\end{aligned}$$

$$\tilde{Q}\,\vec{\rho}\,\tilde{Q}^* = \left[\cos\alpha\,\vec{\rho} + 2\,\mathrm{sen}^2\frac{\alpha}{2}\,(\vec{\rho}\cdot\vec{\nu})\,\vec{\nu} + \mathrm{sen}\,\alpha\,(\vec{\nu}\times\vec{\rho})\right]$$

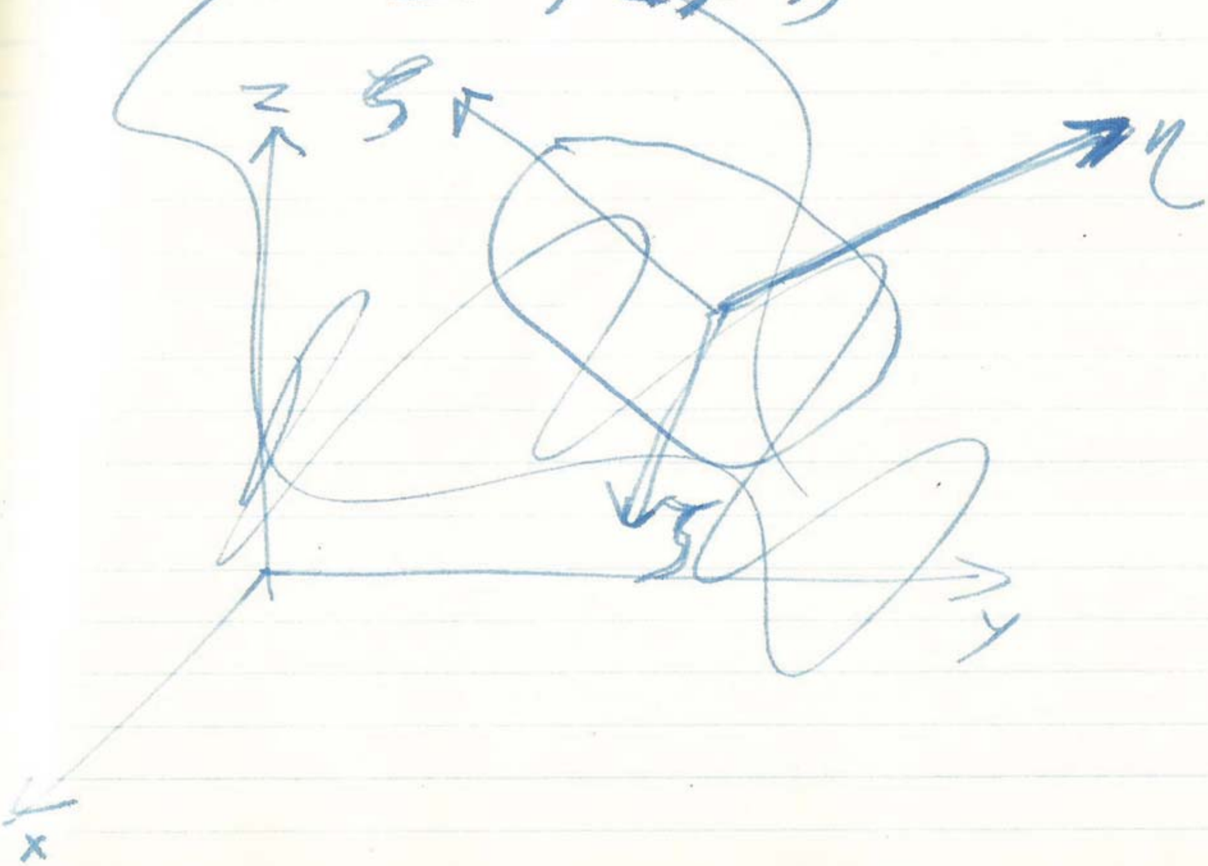
$$\tilde{Q}\,\vec{\rho}\,\tilde{Q}^* = \vec{\rho}\,'$$

$$\boxed{\tilde{Q} = \cos\frac{\alpha}{2} + \vec{\nu}\,\mathrm{sen}\,\frac{\alpha}{2}}$$

$$\boxed{\tilde{Q}^* = \cos\frac{\alpha}{2} - \vec{\nu}\,\mathrm{sen}\,\frac{\alpha}{2}}$$

(X, Y, Z) $\Big\{$ coordenadas de punto fijo en un cuerpo rígido

$$\vec{R} = (X, Y, Z)$$



Principio Variacional.

La variación de la integral $\mathcal{J}$ es nula:

$$\mathcal{J} = \int_a^b \frac{m}{2} e^{\frac{2M}{(\hat{r}\cdot\hat{V})}} \left[2\hat{V} - \frac{\hat{v}}{(\hat{V}\cdot\hat{v})} \right] \cdot d\hat{\rho}$$

$$\delta \mathcal{J} = 0 .$$

Para la notación ver cuaderno α páginas 1, 2, etc.

En el cuaderno β se llegó a la siguiente expresión para

$$\delta \left[e^{\frac{2M}{(\hat{r}\cdot\hat{V})}} \left\{ 2\hat{V} - \frac{\hat{v}}{(\hat{V}\cdot\hat{v})} \right\} \cdot d\hat{\rho} \right]$$

Ver cuaderno β página 25.

$$\delta\left[e^{\frac{2M}{\hat r\cdot\hat V}}\left\{2\hat V-\frac{\hat v}{(\hat V\cdot\hat v)}\right\}\cdot d\hat\rho\right]$$

$$e^{\frac{2M}{(\hat r\cdot\hat V)}}\left[\left(-\frac{4M(\hat r\cdot\hat A)(\hat V\cdot\hat v)}{(\hat r\cdot\hat V)^3}\hat r+\frac{4M(\hat V\cdot\hat v)}{(\hat r\cdot\hat V)^3}\hat r+\frac{2(\hat A\cdot\hat v)}{(\hat r\cdot\hat V)}\hat r+\frac{2M(\hat r\cdot\hat A)}{(\hat r\cdot\hat V)^3(\hat V\cdot\hat v)}\hat r-\frac{2M}{(\hat r\cdot\hat V)^3(\hat V\cdot\hat v)}\hat r+\frac{(\hat A\cdot\hat v)}{(\hat r\cdot\hat V)(\hat V\cdot\hat v)^2}\hat r-\frac{4M(\hat V\cdot\hat v)}{(\hat r\cdot\hat V)^2}\hat V+\frac{2M}{(\hat r\cdot\hat V)^2(\hat V\cdot\hat v)}\hat V\right)\cdot\delta\hat\rho\, ds\right.$$

$$\left.\left(-\frac{2}{(\hat V\cdot\hat v)}\hat v+2\hat V+\frac{1}{(\hat V\cdot\hat v)^2}\hat V\right)\cdot d(\delta\hat\rho)\right]$$

Comprobado $\propto$ cuaderno pág. 11

Comprobado cuaderno $\propto$ pág. 11

$$\underline{\frac{d}{ds}\left[-\frac{2}{(\hat{V}\cdot\hat{v})}\hat{v}\right]}$$

$$\boxed{\frac{d}{ds}\hat{v} = \hat{a}}$$

$$\frac{d}{ds}(\hat{V}\cdot\hat{v}) = \left(\frac{d}{ds}\hat{V}\right)\cdot\hat{v} + \hat{V}\cdot\hat{a}$$

$$\boxed{\frac{d}{ds}\hat{V} = \frac{(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})}\hat{A}}$$

Ver páginas 7 y 12 del cuaderno α.

$$\frac{d}{ds}(\hat{V}\cdot\hat{v}) = \frac{(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})}(\hat{A}\cdot\hat{v}) + (\hat{V}\cdot\hat{a})$$

$$\boxed{\frac{d}{ds}(\hat{V}\cdot\hat{v}) = \frac{(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})} + (\hat{V}\cdot\hat{a})}$$

Cuaderno α página 12

$$\boxed{\frac{d}{ds}\left[\frac{1}{(\hat{V}\cdot\hat{v})}\right] = -\frac{(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{V}\cdot\hat{v})^2(\hat{r}\cdot\hat{V})} - \frac{(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})^2}}$$

cuaderno α pág 13 compr

$$\boxed{\frac{d}{ds}\left[-\frac{2}{(\hat{V}\cdot\hat{v})}\hat{v}\right] = + \frac{2(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{V}\cdot\hat{v})^2(\hat{r}\cdot\hat{V})}\hat{v} + \frac{2(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})^2}\hat{v} - \frac{2}{(\hat{V}\cdot\hat{v})}\hat{a}}$$

comprobado

$$\frac{d}{ds}\left[2\hat{V}\right]$$

$$\frac{d}{ds}\hat{V} = \frac{(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})}\hat{A}$$

$$\boxed{\frac{d}{ds}\hat{V} = \frac{(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})}\hat{A}}$$

Cuaderno α página 12

$$\boxed{\frac{d}{ds}\left[2\hat{V}\right] = 2\frac{(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})}\hat{A}}$$

Comprobado cuaderno α página 12

$$\underline{\frac{d}{ds}\left[\frac{1}{(\hat{V}\cdot\hat{v})^2}\hat{V}\right]}$$

$$\boxed{\frac{d}{ds}\frac{1}{(\hat{V}\cdot\hat{v})^2} = -\frac{2(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{V}\cdot\hat{v})^3(\hat{r}\cdot\hat{V})} - \frac{2(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})^3}}$$

$$\frac{d}{ds}\left[\frac{1}{(\hat{V}\cdot\hat{v})^2}\hat{V}\right] = \left[-\frac{2(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})^3}\hat{V} - \frac{2(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})^3}\hat{V} + \frac{(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})^2}\hat{A}\right]$$

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$$\frac{d}{ds}\left[e^{\frac{2M}{(\hat r\cdot\hat V)}}\right]=$$

$$e^{\frac{2M}{(\hat r\cdot\hat V)}}\left[-\frac{2M(\hat r\cdot\hat A)(\hat r\cdot\hat v)}{(\hat r\cdot\hat V)^3}+\frac{2M(\hat r\cdot\hat v)}{(\hat r\cdot\hat V)^3}-\frac{2M(\hat V\cdot\hat v)}{(\hat r\cdot\hat V)^2}\right]$$

$$\left[-\frac{2}{(\hat V\cdot\hat v)}\hat v+2\hat V+\frac{1}{(\hat V\cdot\hat v)^2}\hat V\right]\rightarrow$$

$$\rightarrow\left[-\frac{2M(\hat r\cdot\hat A)(\hat r\cdot\hat v)}{(\hat r\cdot\hat V)^3}+\frac{2M(\hat r\cdot\hat v)}{(\hat r\cdot\hat V)^3}-\frac{2M(\hat V\cdot\hat v)}{(\hat r\cdot\hat V)^2}\right]$$

$$\left[-\frac{2}{(\hat V\cdot\hat v)}\hat v+2\hat V+\frac{1}{(\hat V\cdot\hat v)^2}\hat V\right]\frac{d}{ds}\left[e^{\frac{2M}{(\hat r\cdot\hat V)}}\right]=$$

$$\left[-\frac{4M(\hat r\cdot\hat A)(\hat r\cdot\hat v)}{(\hat r\cdot\hat V)^3}+\frac{4M(\hat r\cdot\hat v)}{(\hat r\cdot\hat V)^3}-\frac{4M(\hat V\cdot\hat v)}{(\hat r\cdot\hat V)^2}-\frac{2M(\hat r\cdot\hat A)(\hat r\cdot\hat v)}{(\hat r\cdot\hat V)^3(\hat V\cdot\hat v)^2}+\frac{2M(\hat r\cdot\hat v)}{(\hat r\cdot\hat V)^3(\hat V\cdot\hat v)^2}-\frac{2M}{(\hat r\cdot\hat V)^2(\hat V\cdot\hat v)}\right]\hat V$$

$$+\left[\frac{4M(\hat r\cdot\hat A)(\hat r\cdot\hat v)}{(\hat r\cdot\hat V)^3(\hat V\cdot\hat v)}-\frac{4M(\hat r\cdot\hat v)}{(\hat r\cdot\hat V)^3(\hat V\cdot\hat v)}+\frac{4M\,[\text{illegible}]}{(\hat r\cdot\hat V)^2}\right]\hat v$$

Ecuación de las extremales:

$$\left[+ \frac{4M(\hat{r}\cdot\hat{A})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} + \frac{4M(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} + \frac{2(\hat{A}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})} + \frac{2M(\hat{r}\cdot\hat{A})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})} + \frac{2M}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})} + \frac{(\hat{A}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})^2} \right] \hat{r}$$

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$$+ \left[+ \cancel{\frac{4M(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^2}} - \frac{2M}{(\hat{r}\cdot\hat{V})^2(\hat{V}\cdot\hat{v})} - \frac{2(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})^3} - \frac{2(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})^3} - \frac{4M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} + \frac{4M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} - \cancel{\frac{4M(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^2}} - \frac{2M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^2(\hat{V}\cdot\hat{v})^2} + \frac{2M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})^2} - \frac{2M}{(\hat{r}\cdot\hat{V})^2(\hat{V}\cdot\hat{v})} \right] \hat{V}$$

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$$+ \left[+ \frac{2(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{V}\cdot\hat{v})^2(\hat{r}\cdot\hat{V})} + \frac{2(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})^2} + \frac{4M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})} - \frac{4M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})} + \frac{4M}{(\hat{r}\cdot\hat{V})^2} \right] \hat{v}$$

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Ecuación de las extremales del principio variacional de la página 30 expresada como relación entre los 5 cuadrivectores: $\hat{a}$, $\hat{r}$, $\hat{v}$, $\hat{V}$, $\hat{A}$.

$$
\left.
\begin{aligned}
& -\frac{2}{(\hat{V}\cdot\hat{v})}\,\hat{a} \\
& + \left[+ \frac{4M(\hat{r}\cdot\hat{A})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} - \frac{4M(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} - \frac{2(\hat{A}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})} \right. \\
& \qquad \left. - \frac{2M(\hat{r}\cdot\hat{A})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})} + \frac{2M}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})} - \frac{(\hat{A}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})^2} \right] \hat{r} \\
& + \left[+ \frac{2(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{V}\cdot\hat{v})^2(\hat{r}\cdot\hat{V})} + \frac{2(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})^2} + \frac{4M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})} \right. \\
& \qquad \left. - \frac{4M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})} + \frac{4M}{(\hat{r}\cdot\hat{V})^2} \right] \hat{v}
\end{aligned}
\right\} = 0
$$

$$
- \frac{4M}{(\hat{r}\cdot\hat{V})^2(\hat{V}\cdot\hat{v})} - \frac{2(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})^3} - \frac{2(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})^3}
$$

$$
+ \left(- \frac{4M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} + \frac{4M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} - \frac{2M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})^2} + \frac{2M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})^2} \right) \hat{V}
$$

$$
+ \left(+ \frac{2(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})} + \frac{(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})^2} \right) \hat{A}
$$

Ecuación de las extremales del principio variacional de la página 30, expresada de manera que el cuadrivector aceleración esté despejado:

$$\boxed{\hat{a} =} \left[\frac{M}{(\hat{r}\cdot\hat{V})^3} + \frac{2M(\hat{r}\cdot\hat{A})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} - \frac{2M(\hat{V}\cdot\hat{v})^2}{(\hat{r}\cdot\hat{V})^3} - \frac{(\hat{A}\cdot\hat{v})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})} - \frac{M(\hat{r}\cdot\hat{A})}{(\hat{r}\cdot\hat{V})^3} - \frac{\hat{A}\cdot\hat{v}}{2(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})} \right] \hat{r}$$

$$+ \left[\frac{(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{V}\cdot\hat{v})(\hat{r}\cdot\hat{V})} + \frac{(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})} + \frac{2M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} - \frac{2M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} + \frac{2M(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^2} \right] \hat{v}$$

$$+\left[-\frac{2M}{(\hat{r}\cdot\hat{V})^2}-\frac{(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})^2}-\frac{(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})^2}-\frac{2M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3}\right.$$

$$\left.+\frac{2M(\hat{r}\cdot\hat{v})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3}-\frac{M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})}+\frac{M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})}\right]\hat{V}$$

$$+\left[\frac{(\hat{r}\cdot\hat{v})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})}+\frac{(\hat{r}\cdot\hat{v})}{2(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})}\right]\hat{A}$$

$\boxed{\text{Principio variacional}}$ Campo Central

$$-J = \int_A^B \frac{1}{2} e^{\frac{2M}{r}} (1 + v^2)\, dt$$

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \;\|\; r = \sqrt{x^2+y^2+z^2}$$

$$\frac{d}{dt}\left\{e^{\frac{2M}{r}}\dot{x}\right\} = \frac{1}{2} e^{\frac{2M}{r}} \left[-\frac{2M}{r^2}\cdot\frac{x}{r}\right]\{1+v^2\}$$

$$\cancel{e^{\frac{2M}{r}}\ddot{x} + 2\dot{x}\, e^{\frac{2M}{r}}\left[-\frac{2M}{r^2}\cdot\dot{r}\right] = -e^{\frac{2M}{r}}\frac{x}{r^3}}$$

$$\cancel{\ddot{x} = -2\frac{x}{r^3} + 2\frac{\dot{x}\dot{r}}{r^2}}$$

$$e^{\frac{2M}{r}}\ddot{x} - \frac{2M\dot{x}\dot{r}}{r^2} e^{\frac{2M}{r}} = -e^{\frac{2M}{r}}\frac{Mx}{r^3}\{1+v^2\}$$

$$\boxed{\ddot{x} = -\frac{Mx}{r^3} + \frac{2M\dot{x}\dot{r}}{r^2} - \frac{Mxv^2}{r^3}}$$

Ecuaciones de las líneas de universo en el campo central.

$$x'' = -\frac{Mx}{r^3} - \frac{2Mx}{r^3}\left(x'^2+y'^2+z'^2\right) + \frac{Mx'r'}{r^2}$$

$$y'' = -\frac{My}{r^3} - \frac{2My}{r^3}\left(x'^2+y'^2+z'^2\right) + \frac{My'r'}{r^2}$$

$$z'' = -\frac{Mz}{r^3} - \frac{2Mz}{r^3}\left(x'^2+y'^2+z'^2\right) + \frac{Mz'r'}{r^2}$$

$$t't'' - x'x'' - y'y'' - z'z'' = 0$$

$$x'x'' + y'y'' + z'z'' = \left[\begin{array}{l} -\frac{M}{r^3}\left(xx'+yy'+zz'\right) \\ -\frac{2M}{r^3}\left(xx'+yy'+zz'\right)\left(x'^2+y'^2+z'^2\right) \\ +\frac{Mr'}{r^2}\left(x'^2+y'^2+z'^2\right) \end{array}\right]$$

$$x'x'' + y'y'' + z'z'' = \left[\begin{array}{l} -\frac{M}{r^3}\left(xx'+yy'+zz'\right)\left[1+2\left(t'^2-1\right)\right] \\ +\frac{Mr'}{r^2}\left(t'^2-1\right) \end{array}\right]$$

$$x'x'' + y'y'' + z'z'' = -\frac{M}{r^3}\left(2t'^2-1\right)rr' + \frac{Mr'}{r^2}\left(t'^2-1\right)$$

$$x'x'' + y'y'' + z'z'' = -\frac{Mr'}{r^2}\left[2t'^2-1-t'^2+1\right]$$

$$x'x'' + y'y'' + z'z'' = -\frac{Mr't'^2}{r^2}$$

$$t't'' = -\frac{Mr't'^2}{r^2}$$

$$t'' = -\frac{Mr't'}{r^2}$$

$$x'' = -\frac{Mx}{r^3} - \frac{2Mx}{r^3}(x'^2+y'^2+z'^2) + \frac{Mx'r'}{r^2}$$

$$y'' = -\frac{My}{r^3} - \frac{2My}{r^3}(x'^2+y'^2+z'^2) + \frac{My'r'}{r^2}$$

$$z'' = -\frac{Mz}{r^3} - \frac{2Mz}{r^3}(x'^2+y'^2+z'^2) + \frac{Mz'r'}{r^2}$$

$$t'' = \qquad\qquad -\frac{Mr't'}{r^2}$$

$$t't'' - x'x'' - y'y'' - z'z'' =$$

$$\left[+\frac{M}{r^3}(xx'+yy'+zz') + \frac{2M}{r^3}(xx'+yy'+zz')(x'^2+y'^2+z'^2) - \frac{Mr't'^2}{r^2} - \frac{Mr'x'^2}{r^2} - \frac{Mr'y'^2}{r^2} - \frac{Mr'z'^2}{r^2}\right]$$

$$\cancel{= \frac{M}{r^2}\left[rr' + 2rr'(x'^2+y'^2+z'^2) - t'^2 - x'^2 - y'^2 - z'^2\right]}$$

$$t't'' - x'x'' - y'y'' - z'z'' =$$

$$\left[+ \frac{Mr'}{r^2} + \frac{2Mr'}{r^2}(x'^2 + y'^2 + z'^2) - \frac{Mr'}{r^2}(t'^2 + x'^2 + y'^2 + z'^2) \right]$$

$$= \frac{Mr'}{r^2}\left(1 + 2x'^2 + 2y'^2 + 2z'^2 - t'^2 - x'^2 - y'^2 - z'^2\right)$$

$$= \frac{Mr'}{r^2}\left[1 - t'^2 + x'^2 + y'^2 + z'^2\right] = 0.$$

$$\boxed{\begin{aligned}
t'' &= -\frac{Mr't'}{r^2} \\
x'' &= -\frac{Mx}{r^3} - \frac{2Mx}{r^3}(x'^2 + y'^2 + z'^2) + \frac{Mr'x'}{r^2} \\
y'' &= -\frac{My}{r^3} - \frac{2My}{r^3}(x'^2 + y'^2 + z'^2) + \frac{Mr'y'}{r^2} \\
z'' &= -\frac{Mz}{r^3} - \frac{2Mz}{r^3}(x'^2 + y'^2 + z'^2) + \frac{Mr'z'}{r^2}
\end{aligned}}$$

$$\hat{V} = (1, 0, 0, 0);$$

$$\hat{v} = (t', x', y', z');$$

$$\hat{a} = (t'', x'', y'', z'').$$

$$\hat{r} = (t, x, y, z)$$

$$\hat{a} = \left| \begin{array}{l} - \frac{M}{r^3}\hat{r} - \frac{2M}{r^3}\hat{r}(x'^2 + y'^2 + z'^2) + \frac{Mr'}{r^2}\hat{v} \\ + \frac{Mt}{r^3}\hat{V} + \frac{2Mt}{r^3}\hat{V}(x'^2 + y'^2 + z'^2) - 2\frac{Mr't'}{r^2}\hat{V} \end{array} \right.$$

$$\left[\hat{r} - (\hat{r}\cdot\hat{V})\hat{V}\right]^2 = \hat{r}^2 - (\hat{r}\cdot\hat{V})^2$$

$$\hat{r}^2 = t^2 - x^2 - y^2 - z^2$$

$$\hat{r}\cdot\hat{V} = t$$

$$(\hat{r}\cdot\hat{V})^2 = t^2$$

$$\hat{r}^2 - (\hat{r}\cdot\hat{V})^2 = -x^2 - y^2 - z^2$$

$$r^2 = (\hat{r}\cdot\hat{V})^2 - \hat{r}^2$$

$$\boxed{r^3 = \left\{(\hat{r}\cdot\hat{V})^2 - \hat{r}^2\right\}^{3/2}}$$

$$t'^2 - x'^2 - y'^2 - z'^2 = 1$$

$$x'^2 + y'^2 + z'^2 = t'^2 - 1$$

$$\boxed{x'^2 + y'^2 + z'^2 = (\hat{V}\cdot\hat{v})^2 - 1}$$

$$2rr' = 2(\hat{r}\cdot\hat{V})[\hat{V}\cdot\hat{v}] - \cancel{2\hat{r}^2(\hat{v})}\cdot 2(\hat{r}\cdot\hat{v})$$

$$\boxed{rr' = (\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v}) - (\hat{r}\cdot\hat{v})}$$

$$\hat{r}\cdot\hat{V} = t \qquad \hat{V}\cdot\hat{v} = t'$$

$$rr' = \cancel{tt'} - \cancel{tt'} + xx' + yy' + zz'$$

$$\hat{r}\cdot\hat{v} = tt' - xx' - yy' - zz'$$

$$\hat{a} = -\frac{M}{\{(\hat{r}\cdot\hat{V})^2 - \hat{r}^2\}^{3/2}}\hat{r} - \frac{2M[(\hat{V}\cdot\hat{v})^2 - 1]}{\{(\hat{r}\cdot\hat{V})^2 - \hat{r}^2\}^{3/2}}\hat{r}$$

Ecuación diferencial de las líneas de universo en el campo central, en forma invariante.

$$
\begin{aligned}
\hat{a} = & -\frac{M}{\{(\hat{r}\cdot\hat{V})^2-\hat{r}^2\}^{3/2}}\,\hat{r}-\frac{2M[(\hat{V}\cdot\hat{v})^2-1]}{\{(\hat{r}\cdot\hat{V})^2-\hat{r}^2\}^{3/2}}\,\hat{r}\\
& +\frac{M[(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})-(\hat{r}\cdot\hat{v})]}{\{(\hat{r}\cdot\hat{V})^2-\hat{r}^2\}^{3/2}}\,\hat{v}+\frac{M(\hat{V}\cdot\hat{r})}{\{(\hat{r}\cdot\hat{V})^2-\hat{r}^2\}^{3/2}}\,\hat{V}\\
& +\frac{2M(\hat{V}\cdot\hat{r})[(\hat{V}\cdot\hat{v})^2-1]}{\{(\hat{r}\cdot\hat{V})^2-\hat{r}^2\}^{3/2}}\,\hat{V}\\
& -\frac{2M(\hat{V}\cdot\hat{v})[(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})-(\hat{r}\cdot\hat{v})]}{\{(\hat{r}\cdot\hat{V})^2-\hat{r}^2\}^{3/2}}\,\hat{V}
\end{aligned}
$$

¡Comprobado!

Cuadrivector aceleración de una partícula exploradora del campo gravitacional de un punto masa en movimiento ~~acelerado~~ uniforme.

Comprobación:

$$\hat{r} = (t, x, y, z) \qquad \hat{v} = (t', x', y', z')$$

$$\hat{V} = (1, 0, 0, 0)$$

$$(\hat{r}\cdot\hat{V}) = t$$

$$\hat{r}^2 = t^2 - x^2 - y^2 - z^2$$

$$(\hat{r}\cdot\hat{V})^2 - \hat{r}^2 = x^2 + y^2 + z^2 = r^2$$

$$\boxed{\left\{(\hat{r}\cdot\hat{V})^2 - \hat{r}^2\right\}^{3/2} = r^3}$$

$$\cancel{x'' = -\frac{Mx}{r^3}\; 2M}$$

$$(\hat{V}\cdot\hat{v}) = t'$$

$$(\hat{V}\cdot\hat{v})^2 = t'^2$$

$$(\hat{V}\cdot\hat{v})^2 - 1 = t'^2 - 1$$

$$t'^2 - x'^2 - y'^2 - z'^2 = 1$$

$$\boxed{t'^2 - 1 = x'^2 + y'^2 + z'^2}$$

$$\boxed{(\hat{V}\cdot\hat{v})^2 - 1 = x'^2 + y'^2 + z'^2}$$

$$(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v}) - (\hat{r}\cdot\hat{v}) = tt' - tt' + xx' + yy' + zz'$$

$$\boxed{(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v}) - (\hat{r}\cdot\hat{v}) = xx' + yy' + zz'}$$

$$x'' = \left[-\frac{Mx}{r^3} - \frac{2M}{r^3}(x'^2 + y'^2 + z'^2)x + \frac{M(xx' + yy' + zz')}{r^3}x' \right.$$

$$x'' = \left[-\frac{Mx}{r^3} - \frac{2Mx}{r^3}(x'^2 + y'^2 + z'^2) + \frac{Mrr'x'}{r^3} \right]$$

$$t'' = \left[\cancel{-\frac{M}{r^3}t} - \frac{2M(x'^2 + y'^2 + z'^2)}{r^3}t + \frac{Mrr'}{r^3}t' + \cancel{\frac{Mt}{r^3}} + \frac{2Mt}{r^3} - \frac{2Mt'rr'}{r^3} \right]$$

$$t'' = \frac{2Mt}{r^3}\left[1 - x'^2 - y'^2 - z'^2\right]$$

$$t'^2 - (x'^2 + y'^2 + z'^2) = 1$$

$$t'' = \frac{2Mt(t'^2 - 1)}{r^3} = \frac{2Mtt'^2}{r^3} - \frac{2Mt}{r^3}$$

$$L\,dt = \frac{m}{2} e^{\frac{2M}{r}} (1 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2)\, dt$$

$$r = \left\{ (\hat{r} \cdot \hat{V})^2 - \hat{r}^2 \right\}^{1/2}$$

$$\hat{V} = (1, 0, 0, 0)$$

$$\hat{v} = \left(\frac{1}{\sqrt{1-v^2}}, \frac{v^x}{\sqrt{1-v^2}}, \frac{v^y}{\sqrt{1-v^2}}, \frac{v^z}{\sqrt{1-v^2}} \right)$$

$$\hat{V} \cdot \hat{v} = \frac{1}{\sqrt{1-v^2}}$$

$$\frac{1}{(\hat{V} \cdot \hat{v})^2} = 1 - v^2$$

$$\boxed{2 - \frac{1}{(\hat{V} \cdot \hat{v})^2} = 1 + v^2}$$

$$\boxed{dt = \hat{V} \cdot d\hat{r}}$$

$$\hat{V} \cdot (d\hat{\rho} - d\hat{R})$$

$$d\hat{R} = \hat{V} \frac{\hat{r} \cdot \delta\hat{\rho}}{(\hat{r} \cdot \hat{V})}$$

$$dt = \hat{V} \cdot d\hat{\rho} - \frac{\hat{r} \cdot d\hat{\rho}}{\hat{r} \cdot \hat{V}}$$

Ecuación diferencial de las líneas de universo de las partículas exploradoras en el campo central.
Forma invariante.

$$
\begin{aligned}
\hat{a} = {} & \left[+ \frac{M}{\left\{(\hat{r}\cdot\hat{V})^2-(\hat{r})^2\right\}^{3/2}} - \frac{2M(\hat{V}\cdot\hat{v})^2}{\left\{(\hat{r}\cdot\hat{V})^2-(\hat{r})^2\right\}^{3/2}} \right] \hat{r} \\
& + \left[\frac{M(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})}{\left\{(\hat{r}\cdot\hat{V})^2-(\hat{r})^2\right\}^{3/2}} - \frac{M(\hat{r}\cdot\hat{v})}{\left\{(\hat{r}\cdot\hat{V})^2-(\hat{r})^2\right\}^{3/2}} \right] \hat{v} \\
& + \left[- \frac{M(\hat{V}\cdot\hat{r})}{\left\{(\hat{r}\cdot\hat{V})^2-(\hat{r})^2\right\}^{3/2}} + \frac{2M(\hat{V}\cdot\hat{v})(\hat{r}\cdot\hat{v})}{\left\{(\hat{r}\cdot\hat{V})^2-\hat{r}^2\right\}^{3/2}} \right] \hat{V}
\end{aligned}
$$

¡Comprobado!

Cuadrivector aceleración de una partícula exploradora del campo gravitacional de un punto masa en movimiento uniforme.

Masa M en el origen y en reposo.

$$\boxed{\left[\frac{M}{\{(\hat r\cdot\hat V)^2-\hat r^2\}^{3/2}} - \frac{2M(\hat V\cdot\hat v)^2}{\{(\hat r\cdot\hat V)^2-\hat r^2\}^{3/2}}\right] = -\frac{M}{r^3}\{t'^2+x'^2+y'^2+z'^2\}}$$

$$\boxed{\left[\frac{M(\hat r\cdot\hat V)(\hat V\cdot\hat v)}{\{(\hat r\cdot\hat V)^2-\hat r^2\}^{3/2}} - \frac{M(\hat r\cdot\hat v)}{\{(\hat r\cdot\hat V)^2-\hat r^2\}^{3/2}}\right] = \frac{M(xx'+yy'+zz')}{r^3}}$$

$$-\frac{M(\hat V\cdot\hat r)}{\{(\hat r\cdot\hat V)^2-\hat r^2\}} + \frac{2M(\hat V\cdot\hat v)(\hat r\cdot\hat V)}{\{(\hat r\cdot\hat V)^2-\hat r^2\}^{3/2}} =$$

$$\frac{M}{r^3}\left[t(2t'^2-1)-2t'(xx'+yy'+zz')\right]$$

Masa M en reposo en el origen.

$$\hat V=(1,0,0;0)$$

$$\cancel{x''=-(t'^2+x'^2+y'^2+z'^2)x}$$

$$x''=\left[-\frac{M}{r^3}(t'^2+x'^2+y'^2+z'^2)x + \frac{M(xx'+yy'+zz')}{r^3}x'\right]$$

$$t'^2-x'^2-y'^2-z'^2=1$$

$$t'^2=1+x'^2+y'^2+z'^2$$

$$x'' = \left[\begin{aligned} &-\frac{M}{r^3}\left[1+2(x'^2+y'^2+z'^2)\right]x \\ &+\frac{M\,r'x'}{r^2}\end{aligned}\right.$$

$$\boxed{x'' = -\frac{Mx}{r^3} + \frac{2M(x'^2+y'^2+z'^2)}{r^3}\,x + \frac{M r' x'}{r^2}}$$

$$t'' = \left[\begin{aligned} &-\frac{M}{r^3}\{t'^2+x'^2+y'^2+z'^2\}\,t \\ &+\frac{M}{r^3}(xx'+yy'+zz')\,t' \\ &+\frac{M}{r^3}\left[t(2t'^2-1)-2t'(xx'+yy'+zz')\right]\end{aligned}\right]$$

$$t'' = \left[\begin{aligned} &-\frac{Mt}{r^3}\left[1-2t'^2+t'^2+x'^2+y'^2+z'^2\right] \\ &+\frac{Mr't'}{r^2}-\frac{2Mr't'}{r^2}\end{aligned}\right]$$

$$t'' = \left[\begin{aligned} &-\frac{Mt}{r^3}\overbrace{\left[1-t'^2+x'^2+y'^2+z'^2\right]}^{\text{nulo}} \\ &-\frac{Mr't'}{r^2}\end{aligned}\right]$$

$$\boxed{t'' = -\frac{Mr't'}{r^2}}$$

En la ecuación de las extremales de la página 39 hacer $\hat{A}=0$ para obtener las extremales del campo central en forma invariante:

$$\boxed{\begin{aligned}\hat{a} = {} & \left[\frac{M}{(\hat{r}\cdot\hat{V})^3} - \frac{2M(\hat{V}\cdot\hat{v})^2}{(\hat{r}\cdot\hat{V})^3}\right]\hat{r} \\ & + \left[\frac{(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})} - \frac{2M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} + \frac{2M(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^2}\right]\hat{v} \\ & + \left[-\frac{2M}{(\hat{r}\cdot\hat{V})^2} - \frac{(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})^2} + \frac{2M(\hat{r}\cdot\hat{v})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} + \frac{M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})}\right]\hat{V}\end{aligned}}$$

Cálculo de $(\hat{V}\cdot\hat{a})$, $\dfrac{(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})}$ y $\dfrac{\hat{V}\cdot\hat{a}}{(\hat{V}\cdot\hat{v})^2}$

para el campo central. $(\hat{r}^2=0)$

$$\hat{V}\cdot\hat{a}=\left[\cancel{\frac{M(\hat{V}\cdot\hat{r})}{(\hat{V}\cdot\hat{r})^3}}-\frac{2M(\hat{V}\cdot\hat{v})^2(\hat{V}\cdot\hat{r})}{(\hat{V}\cdot\hat{r})^3}\right.$$
$$+\frac{M(\hat{V}\cdot\hat{r})(\hat{V}\cdot\hat{v})^2}{(\hat{V}\cdot\hat{r})^3}-\frac{M(\hat{r}\cdot\hat{v})(\hat{V}\cdot\hat{v})}{(\hat{V}\cdot\hat{r})^3}$$
$$\left.-\cancel{\frac{M(\hat{V}\cdot\hat{r})}{(\hat{V}\cdot\hat{r})^3}}+\frac{2M(\hat{V}\cdot\hat{v})(\hat{r}\cdot\hat{v})}{(\hat{V}\cdot\hat{r})^3}\right]$$

$$\hat{V}\cdot\hat{a}=-\frac{M(\hat{V}\cdot\hat{v})^2}{(\hat{V}\cdot\hat{r})^2}+\frac{M(\hat{r}\cdot\hat{v})(\hat{V}\cdot\hat{v})}{(\hat{V}\cdot\hat{r})^3}$$

$$\boxed{\frac{(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})}=-\frac{M(\hat{V}\cdot\hat{v})}{(\hat{V}\cdot\hat{r})^2}+\frac{M(\hat{r}\cdot\hat{v})}{(\hat{V}\cdot\hat{r})^3}}$$

$$\boxed{\frac{\hat{V}\cdot\hat{a}}{(\hat{V}\cdot\hat{v})^2}=-\frac{M}{(\hat{V}\cdot\hat{r})^2}+\frac{M(\hat{r}\cdot\hat{v})}{(\hat{V}\cdot\hat{r})^3(\hat{V}\cdot\hat{v})}}$$

En la ecuación de las extremales de la página 54 sustituir por $\hat{a}$ la expresión del campo central de la página 51 (habiendo puesto $\hat{r}^2=0$)

$$\hat{a} =$$

$$\left[\frac{M}{(\hat{r}\cdot\hat{V})^3} - \frac{2M(\hat{V}\cdot\hat{v})^2}{(\hat{r}\cdot\hat{V})^3}\right]\hat{r}$$

$$+\left[-\frac{M(\hat{V}\cdot\hat{v})}{(\hat{V}\cdot\hat{r})^2} + \frac{M(\hat{r}\cdot\hat{v})}{(\hat{V}\cdot\hat{r})^3} - \frac{2M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} + \frac{2M(\hat{V}\cdot\hat{v})}{(\hat{V}\cdot\hat{r})^2}\right]\hat{v}$$

$$+\left[-\frac{2M}{(\hat{r}\cdot\hat{V})^2} + \frac{M}{(\hat{r}\cdot\hat{V})^2} - \cancel{\frac{M(\hat{r}\cdot\hat{v})}{(\hat{V}\cdot\hat{r})^3(\hat{V}\cdot\hat{v})}} + \frac{2M(\hat{r}\cdot\hat{v})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} + \cancel{\frac{M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})}}\right]\hat{V}$$

$$\hat{a} =$$

$$\left[\frac{M}{(\hat{r}\cdot\hat{V})^3} - \frac{2M(\hat{V}\cdot\hat{v})^2}{(\hat{r}\cdot\hat{V})^3}\right]\hat{r}$$

$$+\left[+\frac{M(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^2} - \frac{M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3}\right]\hat{v}$$

$$+\left[-\frac{M}{(\hat{r}\cdot\hat{V})^2} + \frac{2M(\hat{r}\cdot\hat{v})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3}\right]\hat{V}$$

Coincide con la $\hat{a}$ de la página 51 para $\hat{r}^2=0$

$$\cancel{\hat{V}\cdot\hat{a}} = \qquad \hat{a} \text{ de la página 39.}$$

$$\frac{M}{(\hat{r}\cdot\hat{V})^2} + \frac{2M(\hat{r}\cdot\hat{A})(\hat{V}\cdot\hat{v})^2}{(\hat{r}\cdot\hat{V})^2} - \cancel{\frac{2M(\hat{V}\cdot\hat{v})^2}{(\hat{r}\cdot\hat{V})^2}} - (\hat{A}\cdot\hat{v})(\hat{V}\cdot\hat{v})$$

$$- \frac{M(\hat{r}\cdot\hat{A})}{(\hat{r}\cdot\hat{V})^2} - \frac{(\hat{A}\cdot\hat{v})}{2(\hat{V}\cdot\hat{v})} + \frac{(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})} + \cancel{(\hat{V}\cdot\hat{a})} + \cancel{\frac{2M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3}}$$

$$- \frac{2M(\hat{r}\cdot\hat{v})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} + \cancel{\frac{2M(\hat{V}\cdot\hat{v})^2}{(\hat{r}\cdot\hat{V})^2}} - \frac{2M}{(\hat{r}\cdot\hat{V})^2} - \frac{(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})^2} - \frac{\hat{V}\cdot\hat{a}}{(\hat{V}\cdot\hat{v})^2}$$

$$- \cancel{\frac{2M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3}} - \frac{M(\hat{r}\cdot\hat{A})^2(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})} + \frac{M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3(\hat{V}\cdot\hat{v})}$$

$$\frac{(\hat{V}\cdot\hat{a})}{(\hat{V}\cdot\hat{v})} =$$

Calcular $\hat{\vec{V}}\cdot\hat{a}$, usando la $\hat{a}$ de la página 39.

$$\frac{2M(\hat{r}\cdot\hat{A})(\hat{\vec{V}}\cdot\hat{v})^3}{(\hat{r}\cdot\hat{\vec{V}})^2} - (\hat{A}\cdot\hat{v})(\hat{\vec{V}}\cdot\hat{v})^2 - \frac{M(\hat{r}\cdot\hat{A})(\hat{\vec{V}}\cdot\hat{v})}{(\hat{r}\cdot\hat{\vec{V}})^2} - \frac{1}{2}(\hat{A}\cdot\hat{v})$$

$$+ \frac{(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})(\hat{\vec{V}}\cdot\hat{v})}{(\hat{r}\cdot\hat{\vec{V}})} - \frac{2M(\hat{r}\cdot\hat{v})(\hat{\vec{V}}\cdot\hat{v})^2}{(\hat{r}\cdot\hat{\vec{V}})^3} - \frac{M(\hat{\vec{V}}\cdot\hat{v})}{(\hat{r}\cdot\hat{\vec{V}})^2}$$

$$- \frac{(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{r}\cdot\hat{\vec{V}})(\hat{\vec{V}}\cdot\hat{v})} - \frac{M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{\vec{V}})^3} + \frac{M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{\vec{V}})^3}$$

Coeficiente de $\hat{v}$ (páginas 39 y 58)

$$\cancel{\frac{(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})}} + \frac{2M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} - \boxed{\frac{2M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3}} + \frac{2M(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^2}$$

$$+ \frac{2M(\hat{r}\cdot\hat{A})(\hat{V}\cdot\hat{v})^3}{(\hat{r}\cdot\hat{V})^2} - (\hat{A}\cdot\hat{v})(\hat{V}\cdot\hat{v})^2 - \frac{M(\hat{r}\cdot\hat{A})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^2} - \frac{1}{2}(\hat{A}\cdot\hat{v})$$

$$+ \frac{(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})(\hat{V}\cdot\hat{v})}{\hat{r}\cdot\hat{V}} - \frac{2M(\hat{r}\cdot\hat{A})(\hat{V}\cdot\hat{v})^2}{(\hat{r}\cdot\hat{V})^3} - \frac{M(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^2}$$

$$- \cancel{\frac{(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})(\hat{V}\cdot\hat{v})}} - \frac{M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3} + \boxed{\frac{M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3}}$$

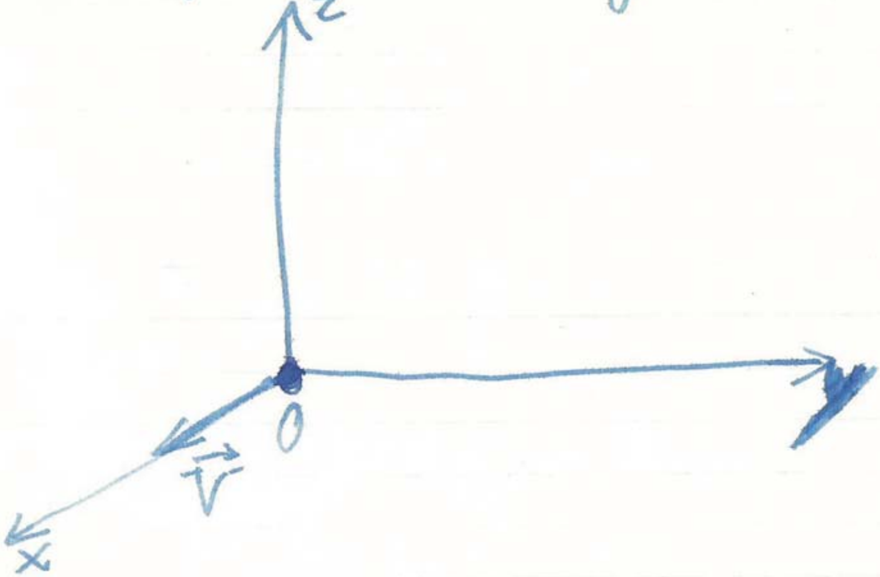
Coeficiente de $\hat{v}$

$$+\frac{M(\hat{r}\cdot\hat{A})(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3}-\frac{M(\hat{r}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^3}+\frac{M(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^2}+\frac{2M(\hat{r}\cdot\hat{A})(\hat{V}\cdot\hat{v})^3}{(\hat{r}\cdot\hat{V})^2}$$

$$-(\hat{A}\cdot\hat{v})(\hat{V}\cdot\hat{v})^2-\frac{1}{2}(\hat{A}\cdot\hat{v})-\frac{M(\hat{r}\cdot\hat{A})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})^2}+\frac{(\hat{r}\cdot\hat{v})(\hat{A}\cdot\hat{v})(\hat{V}\cdot\hat{v})}{(\hat{r}\cdot\hat{V})}$$

$$\cancel{+2M(\hat{r}\cdot\hat{V})}-\frac{2M(\hat{r}\cdot\hat{v})(\hat{V}\cdot\hat{v})^2}{(\hat{r}\cdot\hat{V})^3}$$

Transformación general de Lorentz



$$\bar{x} = \frac{x - Vt}{\sqrt{1-V^2}} \qquad \bar{y} = y \qquad \bar{z} = z$$

$$\bar{t} = \frac{t - Vx}{\sqrt{1-V^2}}$$

$$\bar{x} = x + \frac{-x\sqrt{1-V^2} + x - Vt}{\sqrt{1-V^2}}$$

$$\bar{y} = y$$

$$\bar{z} = z$$

$$\otimes \quad \vec{\bar{R}} = \vec{R} + \frac{\vec{R}\cdot\vec{V}}{V}\left\{\frac{1}{\sqrt{1-V^2}} - 1\right\}\frac{\vec{V}}{V} - \frac{Vt}{\sqrt{1-V^2}}\frac{\vec{V}}{V}$$

$$\vec{\bar{R}} = \vec{R} + \frac{\vec{R}\cdot\vec{V}}{V^2}\left\{\frac{1}{\sqrt{1-V^2}} - 1\right\}\vec{V} - \frac{t}{\sqrt{1-V^2}}\vec{V}$$

$$\vec{\bar{R}} = \vec{R} + \frac{\vec{R}\cdot\vec{V}}{V^2}\left\{\frac{1}{\sqrt{1-V^2}}-1\right\}\vec{V} - \frac{t}{\sqrt{1-V^2}}\vec{V}$$

$$\bar{t} = \frac{t-\vec{V}\cdot\vec{R}}{\sqrt{1-V^2}}$$

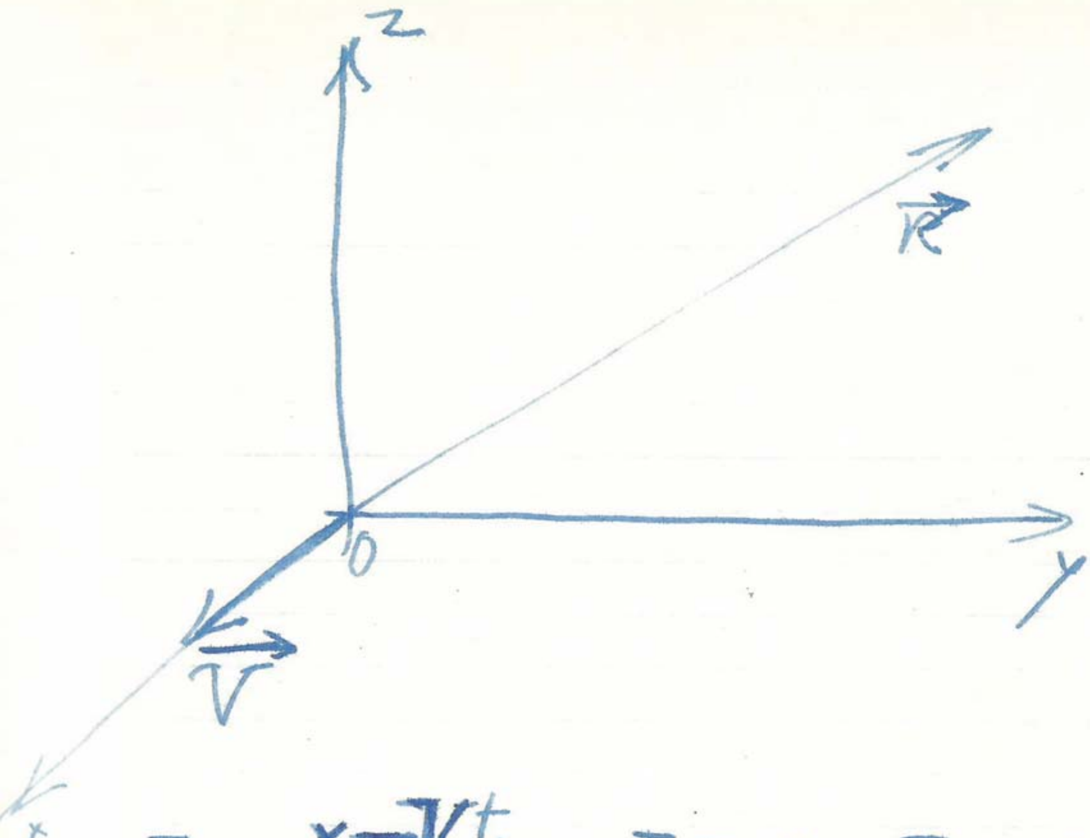
$$\bar{x}^1 = \frac{1}{\sqrt{1-V^2}}x^1 - \frac{V^x}{\sqrt{1-V^2}}x^2 - \frac{V^y}{\sqrt{1-V^2}}x^3 - \frac{V^z}{\sqrt{1-V^2}}x^4$$

$$\cancel{\bar{x}^2 = -\frac{V^x}{\sqrt{1-V^2}}x^1 + \Big\{1+\frac{V^x}{1}}$$

$$\vec{\bar{R}} = \left\{\begin{aligned}&(x,y,z) + \frac{V^x x + V^y y + V^z z}{V^2}\left\{\frac{1}{\sqrt{1-V^2}}-1\right\}(V^x, V^y, V^z)\\ &- \frac{t}{\sqrt{1-V^2}}(V^x, V^y, V^z)\end{aligned}\right\}$$

$$\bar{x} = \left\{\begin{aligned}&x + \frac{V^xV^x x + V^xV^y y + V^xV^z z}{V^2}\left\{\frac{1}{\sqrt{1-V^2}}-1\right\}\\ &- \frac{V^x}{\sqrt{1-V^2}}t\end{aligned}\right.$$

$$\bar{x} = -\frac{V^x}{\sqrt{1-V^2}}x^1 + \Big\{1 + \cancel{\frac{V^x}{\sqrt{1-V^2}}}\, V^xV^x\left\langle\frac{1}{\sqrt{1-V^2}}-1\right\rangle x$$



$$\bar{x} = \frac{x - Vt}{\sqrt{1-V^2}}; \quad \bar{y} = y; \quad \bar{z} = z;$$

$$\bar{t} = \frac{t - Vx}{\sqrt{1-V^2}}$$

$$\vec{R} \cdot \vec{V} = xV = Vx$$

$$\bar{x} = x + \left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\} x - \frac{Vt}{\sqrt{1-V^2}}$$

$$\bar{y} = y$$

$$\bar{z} = z$$

$$\overline{\vec{R}} = \vec{R} + \left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\}\frac{\vec{R}\cdot\vec{V}}{V^2}\vec{V} - \frac{\vec{V}t}{\sqrt{1-V^2}}$$

$$\bar{t} = \frac{t}{\sqrt{1-V^2}} - \frac{\vec{R}\cdot\vec{V}}{\sqrt{1-V^2}}$$

$$\bar{x}^1 = \frac{1}{\sqrt{1-V^2}}x^1 - \frac{V^x}{\sqrt{1-V^2}}x^2 - \frac{V^y}{\sqrt{1-V^2}}x^3 - \frac{V^z}{\sqrt{1-V^4}}x^4$$

$$\boxed{\bar{x}^1 = V^1{}_x{}^1 - V^2{}_x{}^2 - V^3{}_x{}^3 - V^4{}_x{}^4}$$

$$(\bar{x}, \bar{y}, \bar{z}) = \left[(x, y, z) + \left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\}\frac{xV^x + yV^y + zV^z}{V^2}(V^x, V^y, V^z) - \frac{(V^x, V^y, V^z)\,t}{\sqrt{1-V^2}}\right.$$

$$\bar{x} = \left[-\frac{V^x}{\sqrt{1-V^2}}t + \cancel{\left[1 + \left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\}\right.}\,x + x + \left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\}\frac{xV^x + yV^y + zV^z}{V^2}V^x\right]$$

$$\bar{x} = -\frac{V^x}{\sqrt{1-V^2}}t + \left\langle 1 + \left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\}\frac{V^xV^x}{V^2}\right\rangle x$$

$$+ \left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\}\frac{V^xV^y}{V^2}y + \left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\}\frac{V^xV^z}{V^2}z$$

$$\bar{y} = \Big[-\frac{V^y}{\sqrt{1-V^2}}t + \left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\}\frac{V^x V^y}{V^2}x$$

$$+ \left\langle 1 + \left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\}\frac{V^y V^y}{V^2}\right\rangle y$$

$$+ \left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\}\frac{V^y V^z}{V^2}z \Big]$$

$$\& \quad \bar{z} = \Big[-\frac{V^z}{\sqrt{1-V^2}}t + \left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\}\frac{V^x V^z}{V^2}x$$

$$+ \left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\}\frac{V^y V^z}{V^2}y$$

$$+ \left\langle 1 + \left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\}\frac{V^z V^z}{V^2}\right\rangle z \Big]$$

$$\frac{V^x V^z}{V^2}\left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\} = \frac{V^2 V^4}{(V)^2}\left\{-1 + (V)^2 + \sqrt{1-(V)^2}\right\}$$

$$\frac{V^y V^y}{(V)^2}\left\{-1 + \frac{1}{\sqrt{1-V^2}}\right\} = \frac{V^3 V^3}{(V)^2}\left\{-1 + (V)^2 + \sqrt{1-(V)^2}\right\}$$

Matriz

V^{-1}	$-V^{-2}$	$-V^{-3}$	$-V^{-4}$
$-V^{-2}$	$1+\left\{1+(V)^2+\sqrt{1-(V)^2}\right\}\frac{V^{-2}V^{-2}}{(V)^2}$	$\left\{1+(V)^2+\sqrt{1-(V)^2}\right\}\frac{V^{-2}V^{-3}}{(V)^2}$	$\left\{1+(V)^2+\sqrt{1-(V)^2}\right\}\frac{V^{-2}V^{-4}}{(V)^2}$
$-V^{-3}$	$\left\{1+(V)^2+\sqrt{1-(V)^2}\right\}\frac{V^{-2}V^{-3}}{(V)^2}$	$1+\left\{1+(V)^2+\sqrt{1-(V)^2}\right\}\frac{V^{-3}V^{-3}}{(V)^2}$	$\left\{1+(V)^2+\sqrt{1-(V)^2}\right\}\frac{V^{-3}V^{-4}}{(V)^2}$
$-V^{-4}$	$\left\{1+(V)^2+\sqrt{1-(V)^2}\right\}\frac{V^{-2}V^{-4}}{(V)^2}$	$\left\{1+(V)^2+\sqrt{1-(V)^2}\right\}\frac{V^{-3}V^{-4}}{(V)^2}$	$1+\left\{1+(V)^2+\sqrt{1-(V)^2}\right\}\frac{V^{-4}V^{-4}}{(V)^2}$

$$-V^1V^2 + V^2 + \left\{-1 + (V^1)^2 - \boxed{1-(V^1)}\right\}\frac{V^2}{1-(V^1)^2}$$

$$-\cancel{V^1}V^2 + V^2 + V^2 + \cancel{V^1V^2}$$

$$(V^1)^2 - 1$$

$$\cancel{4V^iV^j - 4V^iV^j + 1}$$

$$h^{ij}h_{jk} = \frac{(2V^iV^j - \Delta^{ij})(2V_jV_k - \Delta_{jk})}{(\hat{V}\cdot\hat{r})(\hat{V}\cdot\hat{r})}$$

$$= \frac{4V^i V^j V_j V_k - 2V^iV_k - 2V^iV_k + \delta^i_k}{(\hat{V}\cdot\hat{r})^2}$$

$$= \frac{\delta^i_k}{(\hat{V}\cdot\hat{r})^2}$$

$$\boxed{h^{ij}h_{jk} = \frac{\delta^i_k}{(\hat{V}\cdot\hat{r})^2}}$$ Campo de una ~~masa~~ punto en movim. arbitrario.

$$\begin{Vmatrix} \Phi & -p^x & -p^y & -p^z \\ -p^x & T^{xx} & T^{xy} & T^{xz} \\ -p^y & T^{yx} & T^{yy} & T^{yz} \\ -p^z & T^{zx} & T^{zy} & T^{zz} \end{Vmatrix} \quad \begin{Vmatrix} \Phi & p^x & p^y & p^z \\ p^x & T^{xx} & T^{xy} & T^{xz} \\ p^y & T^{yx} & T^{yy} & T^{yz} \\ p^z & T^{zx} & T^{zy} & T^{zz} \end{Vmatrix}$$

$$\left\|\begin{matrix} \Phi & -p^x & -p^y & -p^z \\ -p^x & T^{xx} & T^{xy} & T^{xz} \\ -p^y & T^{yx} & T^{yy} & T^{yz} \\ -p^z & T^{zx} & T^{zy} & T^{zz} \end{matrix}\right\| \left\|\begin{matrix} \Phi & p^x & p^y & p^z \\ p^x & T^{xx} & T^{xy} & T^{xz} \\ p^y & T^{yx} & T^{yy} & T^{yz} \\ p^z & T^{zx} & T^{zy} & T^{zz} \end{matrix}\right\| =$$

$$\Phi^2 - (p^x)^2 - (p^y)^2 - (p^z)^2$$
$$-\Phi p^x + T^{xx} p^x + T^{xy} p^y + T^{xz} p^z$$
$$-\Phi p^y + T^{yx} p^x + T^{yy} p^y + T^{yz} p^z$$
$$-\Phi p^z + T^{zx} p^x + T^{zy} p^y + T^{zz} p^z$$

$$\Phi p^x - T^{xx} p^x - T^{yx} p^y - p^{zx} p^z$$
$$-(p^x)^2 + T^{xx} + (T^{xy})^2 + (T^{xz})^2$$
$$-p^y p^x + T^{yx} T^{xx} + T^{yy} T^{yx} + T^{yz} T^{zx}$$
$$-p^z p^x + T^{zx} T^{xx} + T^{zy} T^{yx} + T^{zz} T^{zx}$$

$$\Phi p^y - T^{xy} p^x - T^{yy} p^y - T^{yz} p^z$$
$$-p^x p^y + T^{xx} T^{xy} + T^{xy} T^{yy} + T^{xz} T^{zy}$$
$$-p^y p^z + T^{yx} T^{xz} + T^{yy} T^{yz} + T^{yz} T^{zz}$$
$$-p^z p^y + T^{zx} T^{xy} + T^{zy} T^{yy} + T^{zz} T^{zy}$$

$$\Phi p^z - T^{xz} p^x - T^{yz} p^y - T^{zz} p^z$$
$$-p^x p^z + T^{xx} T^{xz} + T^{xy} T^{yz} + T^{xz} T^{zz}$$
$$-p^y p^z + T^{yx} T^{xz} + T^{yy} T^{yz} + T^{yz} T^{zz}$$
$$-(p^z)^2 + T^{zx} T^{xz} + T^{zy} T^{yz} + T^{zz} T^{zz}$$

$$
\begin{array}{llll}
\Phi^2 - \vec{P}^2 & \Phi p^x - (T\vec{P})^x & \Phi p^y - (T\vec{P})^y & \Phi p^z - (T\vec{P})^z \\
-\Phi p^x + (T\vec{P})^x & -p^x p^x + (TT)^{xx} & -p^x p^y + (TT)^{xy} & -p^x p^z + (TT)^{xz} \\
-\Phi p^y + (T\vec{P})^y & -p^y p^x + (TT)^{yx} & -p^y p^y + (TT)^{yy} & -p^y p^z + (TT)^{yz} \\
-\Phi p^z + (T\vec{P})^z & -p^z p^x + (TT)^{zy} & -p^z p^y + (TT)^{zy} & -p^z p^z + (TT)^{zz}
\end{array}
$$

$$\Phi p^x - (T\vec{P})^x = 0$$

$$\Phi \vec{P} - T\vec{P} = 0$$

$$\boxed{T\vec{P} = \Phi \vec{P}}$$

$$\vec{a}\cdot\vec{v} = \Bigg[\frac{d\Phi}{dt} - \frac{\partial\Phi}{\partial t} + 2\frac{d\vec{P}}{dt}\cdot\vec{v} - 2\frac{\partial\vec{P}}{\partial t}\cdot\vec{v}$$

$$+ \left(\frac{dT}{dt}\vec{v}\right)\cdot\vec{v} - \left(\frac{\partial T}{\partial t}\vec{v}\right)\cdot\vec{v} - \vec{v}\cdot\frac{d\vec{P}}{dt}.$$

$$- \left(\frac{dT}{dt}\vec{v}\right)\cdot\vec{v} +$$

$$+ v^2 \Bigg[-\frac{d\Phi}{dt} + \frac{\partial\Phi}{\partial t} + \frac{\partial\vec{P}}{\partial t}\cdot\vec{v} + \left(\frac{\partial T}{\partial t}\vec{v}\right)\cdot\vec{v}$$

$$- \frac{d\vec{P}}{dt}\cdot\vec{v} + \frac{\partial\vec{P}}{\partial t}\vec{v} \Bigg]$$

$$\vec{a} - (\vec{a}\cdot\vec{v})\vec{v} =$$

$$\vec{a} = \Big[-\frac{\partial \vec{P}}{\partial t} + \nabla\left[\Phi + 2(\vec{P}\cdot\vec{v}) + (T\vec{v})\cdot\vec{v}\right]$$

$$+ \dot{\vec{v}}\left[\left(\frac{\partial \vec{P}}{\partial t}\cdot\vec{v}\right) + \left\{\left(\frac{\partial T}{\partial t}\vec{v}\right)\cdot\vec{v}\right\} - \left(\left\{\vec{v}\nabla\vec{P}\right\}\cdot\vec{v}\right) - \frac{\partial T}{\partial t}\right]\vec{v}$$

$$\vec a\cdot\vec v = \dot x\ddot x + \dot y\ddot y + \dot z\ddot z =$$

$$\dot x\frac{\partial\Phi}{\partial x} + \dot y\frac{\partial\Phi}{\partial y} + \dot z\frac{\partial\Phi}{\partial z} - \dot x\frac{\partial P^x}{\partial t} - \dot y\frac{\partial P^y}{\partial t} - \dot z\frac{\partial P^z}{\partial t}$$

$$(\vec a\cdot\vec v)\vec v = \begin{cases} \vec v\Big\langle \vec v\cdot\nabla\Phi + \{2(\vec v\cdot\nabla)\vec P\}\cdot\vec v + \{(\cancel{\vec v\cdot}\nabla)(T)\vec v \\ \quad - \left(\vec v\cdot\frac{d\vec P}{dt}\right)\vec v - \left(\left(\frac{dT}{dt}\vec v\right)\cdot\vec v\right)\Big\rangle\vec v \\ + v^2\vec v\Big[-(\vec v\cdot\nabla)\Phi + \Big\{\frac{\partial P}{\partial t} + \frac{\partial T}{\partial t}\vec v - (\vec v\cdot\nabla)\vec P \end{cases}$$

$$(T\vec v)\cdot\vec v$$

$$T\vec v = \begin{pmatrix} T^{xx}v^x + T^{xy}v^y + T^{xz}v^z \\ T^{yx}v^x + T^{yy}v^y + T^{yz}v^z \\ T^{zx}v^x + T^{zy}v^y + T^{zz}v^z \end{pmatrix}$$

$$(T\vec v)\cdot\vec v = \begin{cases} T^{xx}v^xv^x + \cancel{2}T^{xy}v^xv^y + \cancel{2}T^{xz}v^xv^z \\ + T^{yx}v^xv^y + T^{yy}v^yv^y + T^{yz}v^yv^z \\ + T^{zx}v^xv^z + T^{zy}v^yv^z + T^{zz}v^zv^z \end{cases}$$

$$\frac{\partial}{\partial\vec v}\{(T\vec v)\cdot\vec v\} = \begin{cases} 2\left[T^{xx}v^x + T^{xy}v^y + T^{xz}v^z\right], \\ 2\left[T^{xy}v^x + T^{yy}v^y + T^{yz}v^z\right] \\ 2\left[T^{\cancel{xx}}v^{\cancel{x}} + T^{zy}v^y + T^{zz}v^z\right] \end{cases}$$

$$\frac{\partial}{\partial\vec{v}}\left\{(T\vec{v})\cdot\vec{v}\right\} = 2T\vec{v}.$$

$$\delta\int_a^b e^{\Phi + 2\vec{P}\cdot\vec{v} + (T\cdot\vec{v})\cdot\vec{v}}(1+v^2)\,dt = 0$$

$$\cancel{e^{\Phi + 2\vec{P}\cdot\vec{v} + (T\cdot\vec{v})\cdot\vec{v}}}$$

$$\nabla\left[\Phi + 2\vec{P}\cdot\vec{v} + (T\vec{v})\cdot\vec{v}\right](1+v^2)$$

$$= \frac{d}{dt}$$

$\}\cdot\vec{v}$

$\}\cdot\vec{v}$

$$\ddot{x} + [\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z}]\dot{x}$$

$$\ddot{x} + [\ddot{x}\dot{x}^2 + \ddot{y}\dot{x}\dot{y} + \ddot{z}\dot{x}\dot{z}]$$

$$\left(\frac{\partial \Phi}{\partial x} - \frac{\partial P^x}{\partial t}\right) + \left(\frac{\partial P^x}{\partial x} - \frac{\partial T^{xx}}{\partial t}\right)\dot{x} + \left(2\frac{\partial P^y}{\partial x} - \frac{\partial P^x}{\partial y} - \frac{\partial T^{xy}}{\partial t}\right)\dot{y}$$

$$+ \left(2\frac{\partial P^z}{\partial x} - \frac{\partial P^x}{\partial z} - \frac{\partial T^{xz}}{\partial t}\right)\dot{z}$$

$$+ \left[-\cancel{\frac{\partial \Phi}{\partial x}} + \cancel{\frac{\partial P^x}{\partial t}} + \cancel{\frac{\partial \Phi}{\partial x}} - \cancel{\frac{\partial P^x}{\partial t}}\right]\dot{x}^2$$

ojo

$$+ \left[-\cancel{\frac{\partial \Phi}{\partial y}} + \cancel{\frac{\partial P^y}{\partial t}} + \boxed{\frac{\partial T^{xy}}{\partial x} - \frac{\partial T^{xx}}{\partial y}} + \cancel{\frac{\partial \Phi}{\partial y}} - \cancel{\frac{\partial P^y}{\partial t}}\right]\dot{x}\dot{y}$$

$$+ \left[-\cancel{\frac{\partial \Phi}{\partial z}} + \cancel{\frac{\partial P^z}{\partial t}} + \boxed{\frac{\partial T^{xz}}{\partial x} - \frac{\partial T^{xx}}{\partial z}} + \cancel{\frac{\partial \Phi}{\partial z}} - \cancel{\frac{\partial P^z}{\partial t}}\right]\dot{x}\dot{z}$$

$$+ \left(\frac{\partial T^{yy}}{\partial x} - \frac{\partial T^{xy}}{\partial y}\right)\dot{y}^2 + \left[2\frac{\partial T^{yz}}{\partial x} - \frac{\partial T^{xz}}{\partial y} - \frac{\partial T^{xy}}{\partial z}\right]\dot{y}\dot{z}$$

$$+ \left[\frac{\partial T^{zz}}{\partial x} - \frac{\partial T^{xz}}{\partial z}\right]\dot{z}^2$$

$$+ \left[-\cancel{\frac{\partial P^x}{\partial x}} + \frac{\partial T^{xx}}{\partial t} + \cancel{\frac{\partial P^x}{\partial x}} - \cancel{\frac{\partial T^{xx}}{\partial t}}\right]\dot{x}^3$$

$$+ \left[\begin{array}{l} -\frac{\partial P^x}{\partial y} - \frac{\partial P^y}{\partial x} + 2\frac{\partial T^{xy}}{\partial t} + \cancel{\frac{\partial P^y}{\partial t} - \frac{\partial T^{yy}}{\partial t}} \\ + \frac{2\partial P^x}{\partial y} - \frac{\partial P^y}{\partial x} - \frac{\partial T^{xy}}{\partial t} \\ + 2\frac{\partial P^y}{\partial x} - \frac{\partial P^x}{\partial y} - \frac{\partial T^{xy}}{\partial t} \end{array}\right]\dot{x}^2\dot{y}$$

$$\left[\cancel{-\frac{\partial P^x}{\partial z}} \cancel{- \frac{\partial P^z}{\partial x}} + 2\frac{\partial T^{xz}}{\partial t} \cancel{+ 2\frac{\partial P^z}{\partial x}} \cancel{- \frac{\partial P^x}{\partial z}} - \frac{\partial T^{xz}}{\partial t} \cancel{+ 2\frac{\partial P^x}{\partial t}} \cancel{- \frac{\partial P^z}{\partial x}} - \frac{\partial T^{xz}}{\partial t} \right] \dot{x}^2\dot{z}$$

$$+ \left[\cancel{-\frac{\partial P^y}{\partial y}} \cancel{+ \frac{\partial T^{yy}}{\partial t}} \cancel{+ \frac{\partial P^y}{\partial y}} \cancel{- \frac{\partial T^{yy}}{\partial t}} \right] \dot{x}\dot{y}^2$$

$$+ \left[\cancel{-\frac{\partial P^y}{\partial z}} \cancel{- \frac{\partial P^z}{\partial y}} \cancel{+ 2\frac{\partial T^{yz}}{\partial t}} \cancel{+ 2\frac{\partial P^z}{\partial y}} \cancel{- \frac{\partial P^y}{\partial z}} \cancel{- \frac{\partial T^{yz}}{\partial t}} \cancel{+ 2\frac{\partial P^y}{\partial z}} \cancel{- \frac{\partial P^z}{\partial y}} \cancel{- \frac{\partial T^{yz}}{\partial t}} \right] \dot{x}\dot{y}\dot{z}$$

$$+ \left[\cancel{-\frac{\partial P^z}{\partial z}} \cancel{+ \frac{\partial T^{zz}}{\partial t}} \cancel{+ \frac{\partial P^z}{\partial z}} \cancel{- \frac{\partial T^{zz}}{\partial t}} \right] \dot{x}\dot{z}^2$$

$$+ \left[-\frac{\partial \Phi}{\partial x} + \frac{\partial P^x}{\partial t} \right] \dot{x}^4$$

$$+ \left[-\frac{\partial \Phi}{\partial y} + \frac{\partial P^y}{\partial t} \cancel{+ \frac{\partial T^{xy}}{\partial x}} \cancel{- \frac{\partial T^{xx}}{\partial y}} \cancel{+ \frac{\partial T^{xx}}{\partial y}} \cancel{- \frac{\partial T^{xy}}{\partial x}} \right] \dot{x}^3\dot{y}$$

$$+ \left[-\frac{\partial \Phi}{\partial z} + \frac{\partial P^z}{\partial t} \cancel{+ \frac{\partial T^{xz}}{\partial x}} \cancel{- \frac{\partial T^{xx}}{\partial z}} \cancel{+ \frac{\partial T^{xx}}{\partial z}} \cancel{- \frac{\partial T^{xz}}{\partial x}} \right] \dot{x}^3\dot{z}$$

$$+\left[\cancel{\frac{\partial T^{yy}}{\partial x}}-\cancel{\frac{\partial T^{xy}}{\partial y}}-\frac{\partial \Phi}{\partial x}+\frac{\partial P^{x}}{\partial t}+\cancel{\frac{\partial T^{xy}}{\partial y}}-\cancel{\frac{\partial T^{yy}}{\partial x}}\right]\dot{x}^{2}\dot{y}^{2}$$

$$+\left[2\cancel{\frac{\partial T^{yz}}{\partial x}}-\cancel{\frac{\partial T^{xz}}{\partial y}}-\cancel{\frac{\partial T^{xy}}{\partial z}}+2\cancel{\frac{\partial T^{xz}}{\partial y}}-\cancel{\frac{\partial T^{xy}}{\partial z}}-\cancel{\frac{\partial T^{yz}}{\partial x}}\right.$$
$$\left.+2\cancel{\frac{\partial T^{xy}}{\partial z}}-\cancel{\frac{\partial T^{yz}}{\partial x}}-\cancel{\frac{\partial T^{xz}}{\partial y}}\right]\dot{x}\dot{y}\dot{z}$$

$$+\left[\cancel{\frac{\partial T^{zz}}{\partial x}}-\cancel{\frac{\partial T^{xz}}{\partial z}}-\frac{\partial \Phi}{\partial x}+\frac{\partial P^{x}}{\partial t}+\cancel{\frac{\partial T^{xz}}{\partial z}}-\cancel{\frac{\partial T^{zz}}{\partial x}}\right]\dot{x}^{2}\dot{z}^{2}$$

$$\int \frac{(\vec{v}\cdot\vec{a})\,\vec{v}}{1-v^2}\,dt = \int \vec{v}\,\frac{\vec{v}\cdot\vec{a}}{1-v^2}\,dt$$

$$= \vec{v}\left[-\frac{1}{2}\ln(1-v^2)\right] + \frac{1}{2}\int \ln(1-v^2)\,\vec{a}\,dt$$

$$= -\frac{1}{2}\ln(1-v^2)\vec{v} + \frac{1}{2}\int \ln(1-v^2)\,d\vec{v}$$

$$\cancel{\frac{d}{dt}\frac{v^2}{1-v^2}\vec{v} = \frac{v^2}{1-v^2}\vec{a} + 2\frac{\vec{v}\cdot\vec{a}}{1-v^2}\vec{v}}$$

$$\int \frac{(\vec{a}\cdot\vec{v})(v^2+1)}{1-v^2}\,dt =$$

$$\int\left[\vec{a}\cdot\vec{v} + \frac{2(\vec{a}\cdot\vec{v})}{(1-v^2)}\right]dt$$

$$= \boxed{-\frac{1}{2}v^2 - \ln(1-v^2) + \vec{P}}$$

$$-\frac{1}{6}v^3 -$$

$$\int \ln(1-v^2)\,dv = \int \ln(1+v)\,dv + \int \ln(1-v)\,dv$$

$$\int \ln u\,du \mp \int \ln w\,dw$$

$$U = \int \ln v\, dv = v \ln v - \int \frac{v}{v}\, dv$$

$$U = v \ln v - v$$

$$W = -w \ln w + w$$

$$U = (1+v)\ln(1+v) - (1+v)$$

$$W = -(1-v)\ln(1-v) + (1-v)$$

$$U + W = \left\{ \begin{array}{l} \ln(1+v) - \ln(1-v) + v\left[\ln(1+v) + \ln(1-v)\right] \\ -2v \end{array} \right.$$

$$U + W = \ln\frac{1+v}{1-v} - 2v + v\ln(1-v^2) + C$$

$$\frac{d}{dv}\left[U+W\right] = \left\{ \begin{array}{l} \frac{1}{1+v} + \frac{1}{1-v} - 2 + \ln(1-v^2) \\ - \frac{2v^2}{1-v^2} \end{array} \right\}$$

$$\frac{d}{dv}\left[U+W\right] = \left\{ \begin{array}{l} \frac{\cancel{1} - \cancel{v} + \cancel{1} + \cancel{v} - \cancel{2} + \cancel{2v^2} - 2v^2}{1-v^2} \\ + \ln(1-v^2) \end{array} \right.$$

$$-\frac{1}{6}v^3 + \ln\frac{1-v}{1+v} + 2v - v\ln(1-v^2) + \vec{P}\vec{v}$$

Relación entre Φ, $\vec{P}$ y T en el caso del ~~...~~ campo generado por un punto masa en movimiento arbitrario.

$$h_{ij} = \frac{M[2\hat{V}_i \hat{V}_j - \Delta_{ij}]}{(\hat{V} \cdot \hat{r})}$$

$$\Delta^{ij} h_{ij} = \frac{M[2-4]}{(\hat{V}\cdot\hat{r})} = -\frac{2M}{(\hat{V}\cdot\hat{r})}$$

$$h^{ki} h_{ij} = \frac{M^2[2V^k V^i - \Delta^{ki}][2V_i V_j - \Delta_{ij}]}{(\hat{V}\cdot\hat{r})^2}$$

$$h^{ki} h_{ij} = \frac{M^2}{(\hat{V}\cdot\hat{r})^2}\left[\cancel{4V^k V_j - 2V^k V_j - 2V_j V^k} + \Delta_{ij}\Delta^{ki}\right]$$

$$h^{ki} h_{ij} = \frac{M^2}{(\hat{V}\cdot\hat{r})^2}\delta_j^k$$

$$\begin{pmatrix} \Phi \\ -p^x \\ -p^y \\ -p^z \end{pmatrix}$$

Φ'

$\Phi -$

$$\begin{pmatrix} \Phi & -p^x & -p^y & -p^z \\ -p^x & T^{xx} & T^{xy} & T^{xz} \\ -p^y & T^{yx} & T^{yy} & T^{yz} \\ -p^z & T^{zx} & T^{zz} & T^{zz} \end{pmatrix} \begin{pmatrix} \Phi & p^x & p^y & p^z \\ p^x & T^{xx} & T^{xy} & T^{xz} \\ p^y & T^{yx} & T^{yy} & T^{yz} \\ p^z & T^{zx} & T^{zy} & T^{zz} \end{pmatrix}$$

$$\Phi^2 - P^2 = + \frac{M^2}{(\hat{V} \cdot \hat{r})^2}$$

$$\Phi - T^{xx} - T^{yy} - T^{zz} = - \frac{2M}{(\hat{V} \cdot \hat{r})}$$

Órdenes de Magnitud.

$$x^2 \equiv x \;;\; x^3 \equiv y \;;\; x^4 \equiv z \;;\; x^1 \equiv ct$$

$$\boxed{O(x^\alpha) = 1} \qquad \alpha = 2, 3, 4.$$

$$\boxed{O(x^1) = c} \qquad O(t) = 1$$

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$O(ds^2) = c^2$$

$$\boxed{O(ds) = c}$$

$$\frac{dx^1}{ds},\ \frac{dx^2}{ds},\ \frac{dx^3}{ds},\ \frac{dx^4}{ds}$$

$$\boxed{O\left(\frac{dx^1}{ds}\right) = 1}$$

$$\boxed{O\left(\frac{dx^\alpha}{ds}\right) = \frac{1}{c} = c^{-1}}$$

$$O\left(\frac{d^2x^1}{ds^2}\right) = O\left[\frac{d}{ds}\,\frac{dx^1}{ds}\right] = c^{-1}$$

$$\boxed{O\,\frac{d^2x^\alpha}{ds^2} = c^{-2}} \qquad \boxed{O\,\frac{d^2x^1}{ds} = c^{-1}}$$

Órdenes de Magnitud.

$\boxed{O(x^1) = c}$; $\boxed{O(x^\alpha) = 1}$ Posición

$\boxed{O(v^1) = 1}$ $\boxed{O(v^\alpha) = c^{-1}}$ Velocidad

$\boxed{O(a^1) = c^{-1}}$ $\boxed{O(a^\alpha) = c^{-2}}$ Aceleración

$\boxed{O(ds^2) = c^2}$ $\boxed{O(ds) = c}$ Longitud de arco.

Constante Gravitacional

$$\frac{d^2 x^\alpha}{dt^2} = -\frac{GM}{r^3} x^\alpha$$

$$O\left(\frac{d^2 x^\alpha}{dt^2}\right) = c^{-2} \qquad O(GM) = c^{-2}$$

En unidades relativistas $GM \to M$

$\boxed{O(M) = c^{-2}}$

Caso unidimensional.

$$\ddot{x} = \left[\left(\frac{\partial \Phi}{\partial x} - \frac{\partial P^x}{\partial t}\right) + \left(\frac{\partial P^x}{\partial x} - \frac{\partial T^{xx}}{\partial t}\right)\dot{x} \right.$$

$$\left. \left(-\frac{\partial \Phi}{\partial x} + \frac{\partial P^x}{\partial t}\right)\dot{x}^2 + \left(-\frac{\partial P^x}{\partial x} + \frac{\partial T^{xx}}{\partial t}\right)\dot{x}^3 \right.$$

Cuaderno 6 página 14.

Tres potenciales Φ, $P = P^x$; $T = T^{xx}$

$$\ddot{x} = \left[\left(\frac{\partial \Phi}{\partial x} - \frac{\partial P}{\partial t}\right) + \left(\frac{\partial P}{\partial x} - \frac{\partial T}{\partial t}\right)\dot{x} \right.$$

$$\left. + \left(-\frac{\partial \Phi}{\partial x} + \frac{\partial P}{\partial t}\right)\dot{x}^2 + \left(-\frac{\partial P}{\partial x} + \frac{\partial T}{\partial t}\right)\dot{x}^3 \right]$$

$$\frac{\ddot{x}}{1-\dot{x}^2} = \left(\frac{\partial \Phi}{\partial x} - \frac{\partial P}{\partial t}\right) + \left(\frac{\partial P}{\partial x} - \frac{\partial T}{\partial t}\right)\dot{x}$$

$$\cancel{\frac{d\dot{x}}{dt}} \quad \frac{1}{1-\dot{x}^2} = \frac{\frac{1}{2}}{1-\dot{x}} + \frac{\frac{1}{2}}{1+\dot{x}} = \frac{1}{2}\,\frac{\cancel{1+\dot{x}+1-\dot{x}}}{(1-\dot{x}^2)}$$

$$\frac{1}{2}\,\frac{\ddot{x}}{1-\dot{x}} + \frac{1}{2}\,\frac{\ddot{x}}{1+\dot{x}} = \underline{\frac{d}{dt}\ln\sqrt{\frac{1+\dot{x}}{1-\dot{x}}}}$$

$$\int \left[\frac{1}{2}\ln(1+\dot{x}) - \frac{1}{2}\ln(1-\dot{x})\right] d\dot{x} = ?$$

$$\int \ln(1+\dot{x})\,d\dot{x} = \int \ln(1+v)\,dv$$

$$\int \ln(1+v)\,\underline{dv} = v\ln(1+v) - \int \frac{v}{1+v}\,dv$$

$$= v\ln(1+v) - \int\left[1 - \frac{1}{1+v}\right]dv$$

$$= v\ln(1+v) - v + \ln(1+v)$$

$$= \cancel{v\ln}\ \cancel{(v-1)}\ \underline{\underline{(1+v)\ln(1+v) - v.}}$$

$$\int \ln(1-v)\,dv = v\ln(1-v) + \int \frac{v}{1-v}\,dv$$

$$= v\ln(1-v) + \int\left[-1 + \frac{1}{1-v}\right]dv$$

$$= v\ln(1-v) - v - \ln(1-v)$$

$$= \underline{-(1-v)\ln(1-v) - v}$$

$$\frac{1}{2}\left[(1+v)\ln(1+v) + (1-v)\ln(1-v)\right]$$

$$\boxed{\frac{1}{2}\left[(1+v)\ln(1+v) + (1-v)\ln(1-v)\right]}$$

$$\frac{1}{2}\left[\frac{(1+v)}{1+v} + \ln(1+v) + (1-v)\frac{(-1)}{1-v} - \ln(1-v)\right]$$

$$\frac{1}{2}\left[\ln\frac{1+v}{1-v}\right] = \frac{1}{2}\ln\frac{1+v}{1-v}$$

$$\Phi - T = \sum_\alpha \frac{\cancel{2M}\ 2\ \cancel{-2}}{(\hat{V}^1\cdot\hat{r})_\alpha} = 0$$

$$\Phi = T$$

$$h^{jk} = \left(\frac{2M(V^1V^1 - 1)}{(\hat{V}^1\cdot\hat{r})} - \frac{2MV^1V^2}{(\hat{V}^1\cdot\hat{r})} \right) \qquad \frac{2(MV^1V^1-1)}{(\hat{V}^1\cdot\hat{r})} \quad \frac{2MV^1V^2}{\hat{V}^1\cdot\hat{r}}$$

$$\left(- \frac{2MV^2V^1}{\hat{V}^1\cdot\hat{r}} \qquad \frac{2(MV^2V^2-1)}{(\hat{V}^1\cdot\hat{r})} \right) \qquad \frac{2MV^2V^1}{\hat{V}^1\cdot\hat{r}} \quad \frac{2(MV^2V^2-1)}{\hat{V}^1\cdot\hat{r}}$$

$$h^{jk}h_{ki} = \left(4M^2V^1 \right.$$

$$\frac{4M^2V^1V^1 V^1V^1 - 4MV^1V^1 + 1 - 4M^2V^1V^1V^2V^2}{(\hat{V}^1\cdot\hat{r})^2}$$

$$\frac{4M^2V^1V^1 - 4MV^1V^1 + 1}{(\hat{V}^1\cdot\hat{r})}$$

$$\begin{pmatrix} \Phi & -P \\ -P & T \end{pmatrix} \begin{pmatrix} \Phi & P \\ P & T \end{pmatrix} =$$

$$\Phi^2 - P^2 = 1 \qquad \Phi P - PT = 0$$

$$\boxed{\Phi = T}$$

$$\mathcal{T} = \frac{1}{2}\left[(1+v)\ln(1+v) + (1-v)\ln(1-v)\right]$$ Energía cinética.

$$\mathcal{T} = \frac{1}{2}\left[(1+\dot{x})\ln(1+\dot{x}) + (1-\dot{x})\ln(1-\dot{x})\right]$$

$$\frac{\partial \mathcal{T}}{\partial \dot{x}} = \frac{1}{2}\left[\underbrace{\frac{(1+\dot{x})}{1+\dot{x}}}_{+1} + \ln(1+\dot{x}) + \underbrace{(1-\dot{x})\frac{(-1)}{1-\dot{x}}}_{-1} - \ln(1-\dot{x})\right]$$

$$\frac{\partial \mathcal{T}}{\partial \dot{x}} = \frac{1}{2}\left[\ln(1+\dot{x}) - \ln(1-\dot{x})\right]$$

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{x}} = \frac{1}{2}\left[\frac{\ddot{x}}{1+\dot{x}} + \frac{\ddot{x}}{1-\dot{x}}\right]$$

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{x}} = \frac{1\cancel{-\dot{x}}+1\cancel{+\dot{x}}}{1-\dot{x}^2}\,\frac{\ddot{x}}{2} = \frac{\ddot{x}}{1-\dot{x}^2}$$

$$\mathcal{T} = \frac{1}{2}\left[(1+\dot{x})\ln(1+\dot{x}) + (1-\dot{x})\ln(1-\dot{x})\right]$$

$$L = \frac{1}{2}\left[(1+\dot{x})\ln(1+\dot{x}) + (1-\dot{x})\ln(1-\dot{x})\right] + \Phi + \ldots$$

$$h_{ij} = \begin{pmatrix} \sum_\alpha \frac{2\overset{\alpha}{M}\,\overset{\alpha}{V}{}^1\,\overset{\alpha}{V}{}^1 - 1}{(\hat{V}\cdot\hat{r})_\alpha} & \sum_\alpha \frac{2\overset{\alpha}{M}\,\overset{\alpha}{V}{}^1\,\overset{\alpha}{V}{}^2}{(\hat{V}\cdot\hat{r})_\alpha} \\ \sum_\alpha \frac{2\overset{\alpha}{M}\,\overset{\alpha}{V}{}^1\,\overset{\alpha}{V}{}^2}{(\hat{V}\cdot\hat{r})_\alpha} & \sum_\alpha \frac{2\overset{\alpha}{M}\,\overset{\alpha}{V}{}^2\,\overset{\alpha}{V}{}^2 + 1}{(\hat{V}\cdot\hat{r})_\alpha} \end{pmatrix}$$

$$h_{ij} = \begin{pmatrix} \Phi & P \\ P & T \end{pmatrix}$$

$$\frac{\ddot{x}}{1-\dot{x}^2} = \left(\frac{\partial \Phi}{\partial x} - \frac{\partial P}{\partial t}\right) + \left(\frac{\partial P}{\partial x} - \frac{\partial \Phi}{\partial t}\right)\dot{x}$$

$$\frac{\partial}{\partial \dot{x}}\left[P\dot{x}\right] = P$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left[P\dot{x}\right] = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial x}\dot{x}$$

$$\frac{\partial}{\partial x}\left[P\dot{x}\right] = \frac{\partial P}{\partial x}\dot{x}$$